SYLLABUS

DKM22 – MEASURE THEORY AND COMPLEX ANALYSIS

Unit I: Lebesgue measure – Outer measure – Measurable sets and Lebesgue measure – Measurable functions – Little wood's three principles.

Unit II: Lebesgue integral – The Riemann integral – The lebesgue integral of a bounded function over a set of finite measure – the integral of a non negative function – the general lebesgue integral

Unit III : Complex numbers – Analytic functions – Elementary theory of power series.

Unit IV : Cauchy's Theorem – Cauchy's integral formula – Singularities.

Unit V : Taylor's Theorem – Maximum Principle – The calculus of Residues.

Texts: 1. Royden – Real Analysis Third Edition (PHI) Chapter 3 (Excluding section 3.4), Chapter 4(excluding section 4.5).

2. Ahlfors - Complex Analysis (Tata- McGraw Hill Third Edition) Chapter 1, Chapter 2 (sections 1 and 2) and chapter 4 (sections 1,2,3 and 5).

Unit I

Measure Theory

We need the following definitions and results

Definition :

A Collection $\mathscr{b}$ of subsets of X is called an **algebra** of sets or a **boolien algebra** if,

i. A∪B∈ $\mathscr{b}$ whenever A,B∈ $\mathscr{b}$
ii. A^c is in $\mathscr{b}$ whenever A∈ $\mathscr{b}$

By Demorgan's Law A∩B is in $\mathscr{b}$ whenever A,B∈ $\mathscr{b}$.

Result 1:

Let $\mathscr{b}$ be an algebra of subsets and $\{A_i\}$ a sequence of sets in $\mathscr{b}$.Then there is a sequence $\{B_i\}$ of sets in $\mathscr{b}$, such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$.

Definition:

An algebra $\mathscr{b}$ of sets is called a sigma algebra or a boral field, if every union of a countable collection of sets in $\mathscr{b}$ is again in $\mathscr{b}$.

By Demorgan's Law, the intersection of countable of sets in $\mathscr{b}$ is again in $\mathscr{b}$.

Result 2:

Given any collection ζ of subsets of X, there is a smallest sigma algebra that contains ζ, (i.e) there is a sigma algebra $\mathscr{b}$ containing ζ such that if ẞ is any sigma algebra containing ζ such that $\mathscr{b} \subset$ ẞ.

Definition:

The collection ẞ of **borel sets** is the smallest sigma algebra which contains all of the open sets.

1.1 Lebesgue Measure

Definition:

The length l(I) of the interval I is defined to be the difference of end points of the interval, if I is bounded and if ∞ I is unbounded.

Definition:

A set function m that assigned to each sets E in some collection $\mathcal{M}$ of sets of real numbers a non-negative extended real numbers mE, called **measure of E.**

Properties:

i. mE is defined for each set E of real numbers. (i.e) $M = \mathcal{P}(R)$

ii. For an interval I $\neq \emptyset$, $mI = l(I)$

iii. If $\{E_n\}$ is a sequence of disjoint sets (for which m is defined)

$m(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n)$

iv. m is translation invariant.

(i.e) If E is a set for which m is defined and if E+ y = {x+ y ; x∈ E} obtained by replacing each point x ∈ E by the point x+ y.

Then m (E+ y)= m(E).

Definition:

We say that m is a **count ably additive measure.** If it is a non-negative extended real valued function whose domain of definition is a σ-algebra $\mathcal{M}$ of sets we have m ($\cup E_n$) = $\sum$ m E_n for each $\{E_n\}$ of disjoint sets in $\mathcal{M}$.

Properties:

Let m be a count ably additive measure defined for all sets in $\mathcal{M}$. Then we have the following properties.

i) m(E) ≥ 0, for all E $\in \mathcal{M}$

ii) If A,B $\in \mathcal{M}$ and A C B $\Rightarrow$ m(A) $\leq$ m(B)

Proof:

i) It follows from the definition .

ii) m(A∪B) = m(A) + m(B-A)

$\Rightarrow$ m(B) = m(A) + m(B-A)

$\Rightarrow$m(A)$\leq$ m(B) [since m(B-A) $\geq$ 0]

This property is called **monotonicity.**

iii) Let the collection {E_n} be any sequence of sets in $\mathcal{M}$.

Then $m(\cup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} m(E_n)$. This property is called **countably subadditivity.**

For,

Let {E_n} be a sequence of sets in $\mathcal{M}$. By Result 1, there exists a $\{E_n^{'}\}$ of disjoint sets in $\mathcal{M}$ such that $\cup_{n=1}^{\infty} E_n = \cup_{n=1}^{\infty} E_n^{'}$, where $E_n^{'} = E_n - (E_1 \cup E_2 \cup_{\dots\dots} \cup E_{n-1})$, $E_n^{'} C E_n$

$$m(\cup En) = m(\cup E_n')$$
$$= \sum_{n=1}^{\infty} mE_n^{'}$$
$$\leq \sum_{n=1}^{\infty} mE_n$$

Observation:

If there is a set A $\in \mathscr{B}$ such that mA $< \infty$. Then m$\emptyset$ = 0

For, any set A = A $\cup\ \emptyset$

m(A) = m(A$\cup\ \emptyset$) = m(A) + m($\emptyset$)

$\Rightarrow$ m($\emptyset$) = 0 [m(A) $< \infty$]

Example:

Let $mE = \begin{cases} \infty & , for\ an\ infinte\ set\ E \\ |E| & , for\ a\ finite\ set\ E \end{cases}$

n is countable additive set function and translation invariant. It is defined for all sets of real numbers. This measure is called counting measure.

Solution:

Let $\{E_n\}$ be a sequence of disjoint sets in R. One of the sets, say, E_n is infinite.

Then $n\ (\cup E_n) = |\cup E_n| = \infty$

$= \sum_{i=1}^{\infty} |E_i|$

$= \sum_{n=1}^{\infty} n(E_n)$

If all the sets in $\{E_n\}$are finite and $E_n \cap E_m = \emptyset$ if $n \neq m$.

Then $n(\cup E_n) = |\cup E_n| = \sum_{i=1}^{\infty} |E_i|$

$= \sum_{n=1}^{\infty} n(E_n)$

⇒ n is countably additive.

Also, $n(E + y) = |E + y|$

$= |E|$

$= n(E)$

⇒ n is translation invariant.

1.2 Outer Measure

Definition:

For each set A of real numbers. Consider a countable collection $\{I_n\}$ of open intervals that cover A. (i.e) Collections for which $A \subset \cup I_n$, and for each such collections, consider the sum of the length of the interval in the collection. Then the outer measure m* A to be the infimum of all such sums.

(i.e) $m^*A = \inf_{A \subset \cup I_n} \sum l(I_n)$

Then the immediately the following is observed.

i) $m^*A \geq 0$
ii) $m^*A = mA$, $A \in \mathcal{M}$
(i.e) $m^* = m/\mathcal{M}$

iii) Since $m\emptyset = 0$ we have $m^*\emptyset = 0$

iv) Let $A \subset B$, then $m^*A \leq m^*B$

For, let

$\alpha = \{\{I_n\}/A \subset \cup I_n\}$

$\beta = \{\{I_n\}/B \subset \cup I_n\}$

$\Rightarrow \beta \subset \alpha$

$\Rightarrow \text{Inf} \sum_{\alpha} l(I_n) \leq \inf \sum_{\beta} l(I_n)$

$\Rightarrow \inf \sum_{A \subset \cup I_n} l(I_n) \leq \inf \sum_{B \subset \cup I_n} l(I_n)$

$\Rightarrow m^*A \leq m^*B$

v) $m^*(\{x\}) = 0$

For, $\{x\} \subset (x-\frac{\epsilon}{2}, x+\frac{\epsilon}{2}) = I$

$m^*\{x\} \leq l(I) = \varepsilon$

$\Rightarrow m^*\{x\} = 0$ [since ε is arbitrary]

THEOREM :1

The outer measure of an interval is its length.

Proof:

First we consider the case of closed finite interval, say [a,b].

Now $[a,b] \subset (a-\varepsilon, b+\varepsilon)$

$m^*[a,b] \leq l(a+\varepsilon, b+\varepsilon) = b-a+2\varepsilon$

Since ε is arbitrary, $m^*[a,b] \leq b-a$

To prove: $m^*[a,b] \geq b-a$

Let $[a,b] \subset \bigcup_1^{\infty} I_n$

By Heine Borel theorem, there exists a finite sub collection $I_1, I_2, \ldots\ldots, I_m$ intervals such that $I \subset \bigcup_1^m I_k$ and since the sum of the length of the finite collection is no greater than the sum of the length of the original collection and hence it is enough toprove that $\sum_{k=1}^{n} l\ I_k \geq b-a$ for finite collections $\{I_n\}$ that cover [a, b].

Since a is contained in $\cup I_n$, there must be one of the I_n that contains a.

Let this be the interval (a_1, b_1).

Then we have $a_1 < a < b_1$. If $b_1 < b$, then $b_1 \in (a,b)$

Since $b_1 \notin (a_1,b_1)$, there exists an interval (a_2,b_2) in the collection $\{I_n\}$ such that $b_1 \in (a_2,b_2)$ (i.e) $a_2 < b_1 < b_2$

Continue in this fashion , we obtain a sequence $(a_1,b_1),(a_2,b_2),\ldots\ldots,(a_k,b_k)$ from the collection $\{I_n\}$ such that $a_i < b_{i-1} < b_i$.

Since $\{I_n\}$ is a finite collection, our process must terminate with some finite interval (a_k, b_k).

(i.e) $b \in (a_k , b_k)$ (or) $a_k < b < b_k$.

$\sum_{k=1}^{n} l(I_k) \geq \sum_{k=1}^{n} l(a_i,b_i)$

$= l(a_1,b_1)+\ldots\ldots+ l(a_k,b_k)$

$= b_1- a_1+\ldots\ldots\ldots+b_k- a_k$

$= b_k - (a_k-b_{k-1}) - (a_{k-1}-b_{k-2}) -\ldots\ldots..- (a_2-b_1) - a_1$

$\geq b_k-a_1$ [since $a_i < b_{i-1}$]

As $a_1 < a$ and $b < b_k$

$b_k-a_1 > b-a$

$\sum_{1}^{\infty} l(I_n) > b-a$

By taking inf we have $\inf \sum_{1}^{\infty} l(I_n) \geq b-a$

$m^*[a,b] \geq b-a$

If I is any finite interval then given $\in > 0$ there is a closed interval $J \subset I$ such $l(J) > l(I) - \varepsilon$

Now, $l(I) - \in < l(J) = m^*(J) \leq m^*(I) \leq m^*(\bar{I}) = l(\bar{I}) = l(I)$.

$l(I) - \in < m^*(I) \leq l(I)$

If I is an infinite interval, then given any real number Δ, there is a closed interval $J \subset I$ with $l(J) = \Delta$.

Hence, $m^*(I) \geq m^*(J) = l(J) = \Delta$

Since $m^*(I) \geq \Delta$, for each Δ, we have $m^*(I) = \infty = l(I)$. Hence proved.

THEOREM:2

Let $\{A_n\}$ be a countable collection of sets of real numbers. Then

$m^*(\cup A_n) \leq \sum m^*(A_n)$ [This proposition is called **count ably sub additivity** of m^*].

Proof:

If one of the sets A_n has infinite outer measure then inequality holds trivially.

If m^*A_n is finite then given $\varepsilon > 0$ there is a countable collection $\{I_{n,i}\}_i$ of open intervals such that $A_n \subset \bigcup_i I_{n,i}$ and $\sum l(I_{n,i}) < m^*A_n + 2^{-n}\varepsilon$ (by definition of m^*).

Now the collection $\{I_{n,i}\}_{n,i} = \bigcup_i \{I_{n,i}\}_i$ is countable, being that union of the countable number of countable collection and covers union of A_n.

$$\begin{aligned} m^*(\cup A_n) &\leq \sum_{n,i} l(I_{n,i}) \\ &= \sum_n \sum_i l(I_{n,i}) \\ &\leq \sum_n (m^*A_n + 2^{-n}\varepsilon) \\ &= \sum m^*A_n + \varepsilon \end{aligned}$$

Since ε is arbitrary, $m^*(\cup A_n) \leq \sum m^*A_n$.

COROLLORY:3

If A is countable, then $m^*A=0$

Proof:

Given A is countable. Then $A = \bigcup_{n=1}^{\infty}\{x_n\}$

$$\begin{aligned} m^*A &= m^*\left(\bigcup_{n=1}^{\infty}\{x_n\}\right) \\ &\leq \sum_{n=1}^{\infty} m^*\{x_n\}, \text{ as } m^*\{x_n\} = 0 \\ &= 0 \end{aligned}$$

$m^*A = 0$ (as $m^*A \geq 0$)

COROLLORY:4

The set [0,1] is not countable.

Proof: $m^*[0,1] = l\,[0,1] = 1 \neq 0$

Hence , [0,1] is not countable [by corollary :3]

Definition:

A set which is a countable union of closed sets is called $\boldsymbol{F_\sigma}$.

We say that a set as G_δ if it is the intersection of countable collection of open sets. Note that Complement of G_δ is F_σ and vise versa.

Theorem:5

Given any set A and any $\varepsilon > 0$ there is an open set O such that $A \subset O$ and

$m^*O \leq m^*A + \varepsilon$. There is a $G \in G_\delta$ such that $A \subset G$ and $m^*A = m^*G$.

Proof:

Given $\varepsilon > 0$, by the definition of m^*, there is a countable collection $\{I_n\}$ of open intervals $A \subset \cup I_n$ such that $\sum l(I_n) < m^*A + \epsilon$ -------(1)

Let $O = \cup I_n \Rightarrow O$ is open

$m^*O = m^*\cup I_n$

$\leq \sum m^* I_n$

$= \sum l(I_n)$

$< m^*A + \varepsilon$ ------- (2) (by (1)),

Let $\varepsilon = \frac{1}{n}$

Then by (2), for all n there exists an open set G_n such that $A \subset G_n$ and

$m^*G_n \leq m^*A + \frac{1}{n}$ --------(3)

Let $G = \cap G_n$, then G is a G_δ set and $A \subset G$. $\Rightarrow m^*A \leq m^*G$

Now $A \subset G_n$ $\forall$ n and G_n is open. Also $m^*G \leq m^*G_n \leq m^*A + \frac{1}{n} \forall$ n (by(3))

$\Rightarrow m^*G \leq m^*A$. Hence, $m^*A = m^*G$.

LEMMA:6

If $m^*A=0$ then $m^*(A\cup B) = m^*B$

Proof:

$B \subseteq A\cup B \Rightarrow m^*B \leq m^*(A\cup B) \quad \rightarrow(1)$

By count ably sub additivity property, $m^*(A\cup B) \leq m^*A + m^*B$

Given, $m^*A=0$. Therefore, $m^*(A\cup B) \leq m^*B \quad \rightarrow(2)$

From (1) and (2), $m^*(A\cup B) = m^*B$

1.3 Measurable Sets and Lebesgue Measure

Definition:

A set E is said to be measurable if for each set A, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C)$$

Remark:

(i) Since $A= (A\cap E) \cup (A\cap E^c)$
$\Rightarrow m^*(A) \leq m^*(A\cap E) + m^*(A\cap E^c)$
We have the following definition
E is measurable if for each A we have, $m^*A \geq m^*(A\cap E) + m^*(A\cap E^c)$.

(ii) Since the definition of measurability the symmetric in E and E^c, we have E^c is measurable whenever E is measurable. Clearly, ϕ and R are measurable.

LEMMA:7

If $m^*E=0$ then E is measurable.

Proof:

Let A be any set.

$A \cap E \subset E \Rightarrow m^*(A\cap E) \leq m^*E = 0$

$\Rightarrow m^*(A\cap E) = 0$

Also, $A \cap E^C \subset A \Rightarrow m^*(A\cap E^C) \leq m^*(A)$

Therefore, $m^*(A) \geq m^*(A\cap E^C) + 0$

$\Rightarrow m^*(A) \geq m^*(A \cap E^C) + m^*(A \cap E) \Rightarrow$ E is measurable.

LEMMA:8

If E_1 and E_2 are measurable sets, then $E_1 U E_2$ is measurable.

Proof:

Let A be any set. Since, E_2 is measurable we have,

$m^*\left(A \cap {E_1}^C\right) = m^*\left(A \cap {E_1}^C \cap E_2\right) + m^*(A \cap {E_1}^C \cap {E_2}^C) \longrightarrow$ (1)

Since $A \cap (E_1 U E_2) = (A \cap E_1) \cup (A \cap E_2 \cap E_1^c)$, we have

$m^*(A \cap (E_1 U E_2)) \leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c)$

$\Rightarrow m^*(A \cap (E_1 U E_2)) + m^*(A \cap (E_1^c \cap E_2^c)) \leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c)$
$+ m^*(A \cap (E_1^c \cap E_2^c))$

$= m^*(A \cap E_1) + m^*(A \cap E_1^c)$ (by (1))

$\leq m^*(A)$ [since E_1 is measurable]

Therefore, $m^*(A) \geq m^*((A \cap (E_1 U E_2)) + m^*(A \cap (E_1 \cup E_2)^C)$

$\Rightarrow E_1 U E_2$ is measurable.

LEMMA :9

A family ɱ of measurable sets is an algebra of sets.

Proof:

Let E_1 , $E_2 \in$ ɱ

$\Rightarrow E_1 U E_2$ is measurable (by lemma 8)

$\Rightarrow E_1 U E_2 \in$ ɱ

Also, E ∈ ɱ $\Rightarrow E^C \in$ ɱ (by definition)

Therefore, ɱ is an algebra.

LEMMA : 10

Let A be any set and E_1 , E_2 , …….,E_n be a finite sequence of disjoint measurable sets . Then $m^*(A \cap (\bigcup_{i=1}^{n} E_i)) = \sum_{i=1}^{n} m^*(A \cap E_i)$

Proof:

We prove the lemma by induction on n.

The result is clearly true when n = 1

Assume that the result is true if we have n-1 sets E_i

Since $E_i \cap E_j = \varphi$, $i \neq j$ we have , $(A \cap (\bigcup_{i=1}^{n} E_i)) \cap E_n = A \cap E_n$

$(A \cap (\bigcup_{i=1}^{n} E_i)) \cap E_n^c = (A \cap (\bigcup_{i=1}^{n-1} E_i)) \cap E_n^c$

Since $E_n$is measurable we have,

m^*(A ∩ ($\bigcup_{i=1}^{n} E_i$)) = m^*(A ∩ ($\bigcup_{i=1}^{n} E_i$) ∩ E_n)+m^*(A ∩ ($\bigcup_{i=1}^{n} E_i$) ∩ E_n^c)

= m^*(A ∩ E_n)+ m^*((A ∩ ($\bigcup_{i=1}^{n-1} E_i$)∩ E_n^c)

= m^*(A ∩ E_n)+ m^*(A ∩ ($\bigcup_{i=1}^{n-1} E_i$)

= m^*(A ∩ E_n)+ $\sum_{i=1}^{n-1} m^*$(A ∩ E_i) (by induction hyp)

= $\sum_{i=1}^{n} m^*$(A ∩ E_i)

The theorem is true for all values of n .

THEOREM:11

The collection ɱ of measurable sets is a σ – algebra, that is, complement of a measurable set is measurable , union of a countable collection of measurable sets is measurable. Moreover, every set with outer measure zero is measurable.

Proof:

By lemma 9, ɱ is an algebra of sets.

Claim: ɱ is a $\boldsymbol{\sigma}$ – algebra.

It is enough to prove, E = $\bigcup_{n=1}^{\infty} E_n$, E_n ∈ ɱ ⇒ E ∈ ɱ

Let E = $\bigcup_{n=1}^{\infty} E_n'$, E_n' ∈ ɱ

By a result, we have E = $\bigcup_{i=1}^{\infty} E_i$ and E_j ∩ E_i = φ, $i \neq j$ and

$\bigcup_{i=1}^{\infty} E_i$= $\bigcup_{i=1}^{\infty} E_i'$

Let A be any set. Let F_n= $\bigcup_{i=1}^{n} E_i$

⇒ F_n is measurable (i.e) F_n Є ɱ

Now , $F_n = \bigcup_{i=1}^{n} E_i \subset \bigcup_{i=1}^{\infty} E_i =$ E

=> F_n ⊂ E , for all n => $F_n^c \supset E^c$, for all n

Since F_n is measurable ,

m^*(A) = m^*(A ∩ E_n) + m^*(A ∩ E_n^c)

≥ m^*(A ∩ E_n) + m^*(A ∩ E^c)

m^*(A ∩ E_n) = m^*(A ∩ $\bigcup_{i=1}^{n} E_i$)

= $\sum_{i=1}^{n} m^*$(A ∩ E_i) (by lemma 10)

Therefore , m^*(A) ≥ $\sum_{i=1}^{n} m^*$(A ∩ E_i) + m^*(A ∩ E^c)

This is true for every n and L.H.S is independent of n.

We have , m^*(A) ≥ $\sum_{i=1}^{\infty} m^*$(A ∩ E_i) + m^*(A ∩ E^c)

≥ m^*[U (A ∩ E_i)]+ m^*(A ∩ E^c) (by countably subadditive property)

$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$

$\Rightarrow$ E is measurable

Therefore , ɱ is σ–algebra

For, Intersection of countable collection of sets is measurable.

Let $E_n \in$ ɱ $\Rightarrow E_n^c \in$ ɱ $\Rightarrow \bigcup_{n=1}^{\infty} E_n^c \in$ ɱ

$\Rightarrow (\bigcup_{n=1}^{\infty} E_n{}^C)^C \in$ ɱ $\Rightarrow \cap E_n \in$ ɱ

Also by lemma 7, every set with outer measure zero is measurable.

LEMMA: 12

Open interval (a,∞) is measurable.

Proof:

Let A be any set

Let A_1 = (a,∞) , A_2 = A ∩ (-∞,a]

To prove (a, ∞) is measurable.

Claim: $m^*(A) \geq m^*(A_1) + m^*(A_2)$

If $m^*(A) = \infty$ then there is nothing to prove

If $m^*(A) < \infty$,then given $\epsilon>0$, there exists a countable collection of open intervals $\{I_n\}$ which covers A and for which $\sum l(I_n) < m^*(A) + \epsilon \rightarrow (1)$ by the definition of outer measure .

Let $I_n' = I_n \cap (a,\infty)$ and $I_n'' = I_n \cap (-\infty, a]$

Then I_n' and I_n'' are intervals (or) empty.

Now , $l(I_n) = l(I_n') + l(I_n'') = m^*(I_n') + m^*(I_n'') \rightarrow (2)$

Since $A_1 \subset \cup I_n'$,

$m^*(A_1) \leq m^*(\cup I_n') \leq \sum m^*(I_n')$

Similarly, $A_2 \subset \cup I_n''$,

$m^*(A_2) \leq m^*(\cup I_n'') \leq \sum m^*(I_n'')$

Therefore, $m^*(A_1) + m^*(A_2) \leq \sum m^*(I_n') + \sum m^*(I_n'')$

$= \sum (m^*(I_n') + \sum m^*(I_n''))$

$= \sum l(I_n)$ (by (2))

$< m^*(A) + \varepsilon$ (by (1))

Since, ε is arbitrary , $m^*(A_1) + m^*(A_2) < m^*(A)$.

THEOREM:13

Every Borel set is measurable. In particular , each open set and each closed set is measurable.

Proof:

Let Ɓ be the family of Borel sets.

By definition ,Ɓ is the smallest σ – algebra containing all the open sets →(1)

Also ɱ, the collection of all measurable sets is σ – algebra.

Since (a,∞) Є ɱ, ⇒ $(a,\infty)^c$ Є ɱ , for all a ⇒ $(-\infty, a]$ Є ɱ , for all a

Now , $(-\infty,b) = \bigcup_{n=1}^{\infty}\left(-\infty, b-\frac{1}{n}\right]$

⇒ $\bigcup_{n=1}^{\infty}\left(-\infty, b-\frac{1}{n}\right]$ Є ɱ [as ɱ is a σ – algebra]

⇒ (-∞, b) Є ɱ

Now , (a, b) = (-∞,b) ∩ (a,∞)

Therefore , every open interval is measurable

Since each open set is the countable union of open intervals , every open set belongs to ɱ. Then every closed set is measurable. Therefore, ɱ is a σ-algebra containing all the open sets.Therefore, Ɓ ⊂ ɱ [by 1]. Hence the result.

Remark:

If E is measurable set, we define the lebesgue measure mE be the outer measure of E. (ie.) m= m*/ ɱ. It is means that the domain of m is ɱ and the domain of m* is $\mathcal{P}$(R). (ie.) If E is measurable set, mE= m*E.

THEOREM :14

Let $\{E_i\}$ be a sequence of measurable sets, then $m(\cup E_i) \leq \sum mE_i$. If the sets $\{E_n\}$ are pairwise disjoint then $m(\cup E_i) = \sum mE_i$.

Proof:

If $\{E_i\}$ is a finite sequence of disjoint measurable sets, then by lemma 10 (by taking A = R) we have, $m^*(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m^*E_i$.

$\Longrightarrow$ $m(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} mE_i$ and so m is finitely additive.

Let $\{E_n\}$ be an infinite sequence of pair wise disjoint measurable sets.

As $\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^{n} E_i \ \forall n$

$m(\bigcup_{i=1}^{\infty} E_i) \geq m(\bigcup_{i=1}^{n} E_i)$

$= \sum_{i=1}^{n} mE_i$ is true for every n.

Since Left hand side of inequality is independent of n, We have,

$m(\bigcup_{i=1}^{\infty} E_i) \geq \sum_{i=1}^{\infty} m\,E_i$.

The reverse inequality follows from the countable sub additive property,

$m(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} mE_i$. Therefore, $m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m\, E_i$.

Theorem:15

Let $\{E_n\}$ be an infinite decreasing sequence of measurable sets, that is with $E_{n+1} \subset E_n$, for each n. Let mE_1 be finite and $m(\bigcap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} mE_n$.

Proof:

Let $E = \bigcap_{i=1}^{\infty} E_i$. Let $F_i = E_i \sim E_{i+1}$.

Then $E_1 - E = \bigcup_{i=1}^{\infty} F_i$ and the set F_i are pair wise disjoint.

Hence $m(E_1 \sim E) = m(\bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} mF_i = \sum_{i=1}^{\infty} m(E_i \sim E_{i+1}) \rightarrow (1)$

Since $E \subset E_1$, $E_1 = E \cup (E_1 \sim E)$ and $mE_1 = mE + m(E_1 \sim E) \rightarrow (2)$

Similarly, Since $E_{i+1} \subset E_i \implies E_i = E_{i+1} \cup (E_i \sim E_{i+1})$ and

$mE_i = mE_{i+1} + m(E_i \sim E_{i+1})$. Also $mE \leq mE_1 < \infty$

$\Rightarrow m(E_1 - E) = mE_1 - mE$ [by (1)].

And $m(E_i \sim E_{i+1}) = mE_i - mE_{i+1}$ [since, $E_{i+1} \subset E_i \subset E_1$ and $mE_{i+1} \leq mE_1 < \infty$]

$$\begin{aligned}
\text{Therefore, } mE_1 - mE &= m(E_1 \sim E) \\
&= \sum_{i=1}^{\infty} m(E_i \sim E_{i+1}) \quad \text{[by (1)]} \\
&= \sum_{i=1}^{\infty} (mE_i - mE_{i+1}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n-1} (mE_i - mE_{i+1}) \\
&= \lim_{n\to\infty} (mE_1 - mE_n)
\end{aligned}$$

$$mE_1 - mE = mE_1 - \lim_{n\to\infty} mE_n$$

$$mE = \lim_{n\to\infty} mE_n \quad \text{[since, } mE_1 < \infty]$$

$$m(\bigcap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} mE_n \ .$$

THEOREM:16

Let E be the given set, then the following are equivalent

i) E is measurable.

ii) given $\varepsilon > 0$ there is an open set O contains E with $m^*(O-E) < \varepsilon$

iii) given $\varepsilon > 0$ there is a closed set F contained in E with $m^*(E-F) < \varepsilon$

iv) there is a G in G_δ with $E \subset G$ and $m^*(G-E)=0$

v) there is a F in F_σ with $F \subset E$ such that $m^*(E-F) = 0$

m^*E is finite ,then the above statements are equivalent to (vi).

vi) given $\varepsilon > 0$,there is a finite union of open intervals such that $m^*(U \Delta E) < \varepsilon$

Proof:

We prove the theorem as follows:

(i) => (ii)=> (iv)=>(i)

(ii) => (iii)=> (v)=>(i) and (i) ⇔(vi)

step I:

(i) => (ii)

Given E is measurable.

Case (i):

Suppose $m^*E = mE < \infty$ with $m^*(E-F_n) < \frac{1}{n}$

⇨ Given $\varepsilon > 0$, there exists a collection $\{I_n\}$ of open intervals such that $E \subset \cup I_n$ and $\sum l(I_n) \leq m^*E + \varepsilon$.

Let $O = \cup I_n$. Then $mO = m(\cup I_n) \leq \sum mI_n = \sum l(I_n)$

$$mO \leq m^*E + \varepsilon \quad \rightarrow (1)$$

Now $E \subset \cup I_n = O$

$\Rightarrow m^*O = m^*E + m^*(O-E)$

$\Rightarrow m^*(O-E) = m^*O - m^*E$ [since mE is finite]

$\Rightarrow m^*(O-E) \leq m^*E + \varepsilon - m^*E$

$= \varepsilon$ [since $mO = m^*O$ and O is measurable]

$\Rightarrow m^*(O-E) < \varepsilon$

Case(ii):

Let m^*E is infinite

Let $E \subset \bigcup_{n=1}^{\infty} I_n$, where $\{I_n\}$ is a collection of intervals of finite length.

Define $E_n = E \cap I_n \Rightarrow E = \bigcup_{n=1}^{\infty} E_n$

As E, I_n are measurable , $E \cap I_n$ is measurable. ⇒ E_n is measurable for all n.

Now $E_n \subset I_n \Rightarrow m^*E_n < m^*I_n < \infty \Rightarrow m^*E_n$ is finite for all n.

By case(i), for given $\varepsilon > 0$, there exists an open set G_n such that $E_n \subset G_n$, n = 1,2,3,…. and $m^*(G_n - E_n) < \varepsilon/2^n$

Let $O = \bigcup_{n=1}^{\infty} G_n \Rightarrow$ O is open

Consider $O - E = \bigcup_{n=1}^{\infty} G_n - \bigcup_{n=1}^{\infty} E_n$

$\subset \bigcup_{n=1}^{\infty}(G_n - E_n)$

$m^*(O - E) \leq m^*(\bigcup_{n=1}^{\infty}(G_n - E_n))$

$\leq \sum_{n=1}^{\infty} m^*(G_n - E_n)$

$< \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \varepsilon \Rightarrow m^*(O - E) < \varepsilon$

(ii) => (iv):

Given $\varepsilon > 0$, there exists an open set O with $E \subset O$ such that $m^*(O-E) < \varepsilon$

For each n, taking $\varepsilon = 1/n$ we get an open set O_n such that $E \subset O_n$ with $m^*(O_n-E) < 1/n \rightarrow (1)$

Let $G = \cap O_n$, then $G \in G_\delta$

Since $E \subset O_n, \forall n \Rightarrow G-E \subset O_n-E, \forall n$

$\Rightarrow m^*(G-E) \leq m^*(O_n-E) < 1/n, \forall n$

Therefore, $m^*(G\text{-}E) = 0$

(iv)=>(i):

There is a G in G_δ with $E \subset G$ and $m^*(G\text{-}E) = 0$

Since each open set is measurable, each G_δ set is measurable.

Therefore, G is measurable.

Also $m^*(G\text{-}E)=0$,then by lemma 7, G-E is measurable.

But $E = G-(G-E)$ and hence E is measurable.

Step II:

(ii) =>(iii):

Now (i) => (ii) follows from step I

$\Rightarrow$ E is measurable $\Rightarrow$ E^c is measurable.

Given $\varepsilon > 0$, there exist an open set $O \supset E^c$ such that $m^*(O - E^c) < \varepsilon$

$\Rightarrow$ $O^c \subset E$

Since O is open, $F = O^c$ is closed.

Now $F \subset E$ and $m^*(E\text{-}F) = m^*(E\text{-}O^c) = m^*(O\text{-}E^c) < \varepsilon$

(iii)=> (v):

Given $\varepsilon > 0$, there exists a closed set C such that $C \subset E$ and $m^*(E\text{-} C) < \varepsilon$

For each n, there exists a closed set F_n such that $F_n \subset E$ with $m^*(E\text{-} F_n) < \frac{1}{n}$

Let $F = \cup F_n$. F is a F_σ set

Now, $F_n \subset E$ for all n. Then $F \subset E$

Since $F_n \subset F \Rightarrow E\text{-}F \subset E - F_n$

$m^*(E\text{-}F) \leq m^*(E\text{-} F_n) < \frac{1}{n}$, for all n $\Rightarrow$ $m^*(E\text{-}F) = 0$

(v) => (i) :

Given, there exists a F_σ - set $F \subset E$ such that $m^*(E\text{-}F)=0$

Since F_σ set is measurable, F is measurable.

Also, $m^*(E\text{-}F)=0$, by lemma7, E-F is measurable.

But, $E = (E\text{-}F) \cup F$. Since F is measurable, E is measurable

Step III:

(i)$\Leftrightarrow$(vi)

To Prove (i) => (vi)

Suppose E is measurable.

Given $\varepsilon > 0$ there exists an open set O ⊃ E such that $m^*(O-E) < \frac{\varepsilon}{2}$ ---(1) (by (ii))

As mE is finite ((i.e) mE < ∞) and m(O) = mE + m(O-E) < ∞

As O is an open set, O is the disjoint union of open intervals I_i, (i.e) $O=\bigcup_{i=1}^{\infty} I_i$, $I_i \cap I_j=\emptyset$ ($i\neq j$). Therefore, $m(O) = m(\bigcup_{i=1}^{\infty} I_i) = \sum_{i=1}^{\infty} mI_i = \sum_{i=1}^{\infty} l(I_i)$

Now, $m(O) < \infty \Rightarrow$ There exists n such that $\sum_{i=n+1}^{\infty} l(I_i) < \varepsilon/2$

Let $U = \bigcup_{i=1}^{n} I_i$

$E\Delta U = (E-U) \cup (U-E)$

$E\Delta U \subset (O-U) \cup (O-E)$ [∵ E⊂O & O⊂U]

$m^*(E\Delta U) \leq m^*(O-U)+m^*(O-E)$ [∵ E⊂O & O⊂U]

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ [$O-U=\sum_{i=1}^{\infty} I_i - \sum_{i=1}^{n} I_i =\sum_{i=n+1}^{\infty} I_i$

$m^*(O-U)\leq \sum_{i=n+1}^{\infty} l(I_i)$, by definition]

Therefore, $m^*(E\Delta U) < \varepsilon$

(vi) => (i):

Conversely, suppose given $\varepsilon > 0$, there exists intervals $\{I_i\}_{i=1}^{n}$ such that $U=\bigcup_{i=1}^{n} I_i$ and $m^*(E\Delta U) < \varepsilon$

Let $\varepsilon > 0$ be given and also given $m^*E < \infty$

Therefore, there exists an open set O ⊃ E such that $m^*O < m^*E + \varepsilon/3$-----(1) [by Theorem 5]

Also given there exists intervals $\{I_i\}_{i=1}^{n}$ such that $m^*(E\Delta U) < \varepsilon/3$ ----------(2),

where $U= \bigcup_{i=1}^{n} I_i$

Let J = U ∩ O . Then J is open

Also, $J\Delta E = (J\cup E) - (J\cap E)$

Now, $J\cap E = (U\cap O) \cap E = U \cap (O\cap E) = U \cap E$ [∵ E ⊂ O]

$J\cup E \subset U\cup E \Rightarrow J\Delta E \subset U\Delta E$.

$m^*(J\Delta E) \leq m^*(U\Delta E) < \varepsilon/3$ ---------- (3) (by (2))

$O\Delta E \subset (O\Delta J) \cup (J\Delta E)$ [as $O-E\subset (O-J)\cup(J-E)$

and $E-O\subset (E-J)\cup(J-O)$]

$m^*(O\Delta E) \leq m^*(O\Delta J) + m^*(J\Delta E)$

But $O \Delta J = O - J$ [$\therefore O \supset J$]

Therefore, $m^*(O \Delta J) = m^*(O - J)$

$= m^*(O) - m^*(J)$ [$\therefore$ O, J open => O, J are measurable]

$\longrightarrow$ (4)

But $E \subset J \cup (E-J) \Rightarrow E \subset J \cup (E\Delta J)$

$m^*E \leq m^*J + m^*(E \Delta J) \Rightarrow m^*E < m^*J + \frac{\varepsilon}{3}$ (by (3)) $\longrightarrow$ (5)

Now, $m^*(O\Delta E) \leq m^*(O\Delta J) + m^*(J\Delta E)$

$= m^*(O) - m^*(J) + m^*(J \Delta E)$ (by(4))

$\leq m^*E + \frac{\varepsilon}{3} - m^*J + \frac{\varepsilon}{3}$ (by(1) & (3))

$\leq m^*J + \frac{\varepsilon}{3} - m^*J + \frac{\varepsilon}{3}$ (by(5))

$= \varepsilon$

$m^*(O\Delta E) < \varepsilon$

Since $E \subset O \Rightarrow O \Delta E = O - E$

$\therefore m^*(O - E) < \varepsilon$. This proves (ii)

By step II, (ii) => (i) (i.e.) E is measurable.

1.4 Measurable Functions

LEMMA:17

Let f be an extended real valued function whose domain is measurable. Then the following statements are equivalent

(i) For each real number α, the set $\{x/ f(x)>\alpha\}$ is measurable.
(ii) For each real number α, the set $\{x/ f(x)\geq \alpha\}$ is measurable.

(iii) For each real number α, the set $\{x / f(x) < \alpha\}$ is measurable.

(iv) For each real number α, the set $\{x / f(x) \leq \alpha\}$ is measurable.
These statements imply (v)

(v) For each extended real number α, the set $\{x/f(x) = \alpha\}$ is measurable.

Proof:

Let D = the domain of f . D is measurable (given)

$(\boldsymbol{i}) \Rightarrow (iv)$:

Given : $\{x/f(x) > \alpha\}$ is measurable for all α

Now, $\{x/f(x) \leq \alpha\} = D - \{x / f(x) > \alpha\}$. Since, the difference between two measurable sets is measurable we have $\{x / f(x) \leq \alpha\}$ is measurable.

$(\boldsymbol{iii}) \Rightarrow (ii)$:

Suppose $\{x / f(x) < \alpha\}$ is measurable.

Now $\{x/f(x) \geq \alpha\} = D \sim \{x/f(x) < \alpha\}$. Since D is measurable, we have

$\{x/f(x) \geq \alpha\} = D \sim \{x/f(x) < \alpha\}$ is measurable.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{iii})$

Suppose $\{x/f(x) \geq \alpha\}$ is measurable.

Now $\{x/f(x) < \alpha\} = D \sim \{x / f(x) \geq \alpha\}$

Since the difference of two measurable sets is measurable

Therefore $\{x / f(x) < \alpha\}$ is measurable.

Hence, we have proved $(\boldsymbol{i}) \Rightarrow (\boldsymbol{iv})$ **,** $(\boldsymbol{ii}) \Leftrightarrow (\boldsymbol{iii})$

Similarly, we can prove $(\boldsymbol{iv}) \Rightarrow (\boldsymbol{i})$**.**

$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$

Since $\{x/f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x/f(x) > \alpha - \frac{1}{n}\}$ and the intersection of a sequence of measurable sets is measurable

Therefore , $\{x / f(x) \geq \alpha\}$ is measurable.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$

$\{\, x \,/ f(x) > \alpha\} = \bigcup_{n=1}^{\infty}\{x/\, f(x) \ \geq \alpha + \frac{1}{n}\ \}$ and the union of a sequence of measurable sets is measurable. Therefore, $\{\, x \,/ f(x) > \alpha\}$ is measurable.

Hence all the above four statements are equivalent.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{v})$

If α is real, $\{x \,/ f(x) = \alpha\} = \{x \,/\, f(x) \geq \alpha\} \cap \{x \,/\, f(x) \leq \alpha\}$

Therefore, $(ii)\ and\ (iv)$ together implies (v)

Since $\{x \,/ f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \,/\, f(x) \geq n\}$

By (ii), $\{x \,/ f(x) \geq n\}$ $is\ measurable.$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x \,/\, f(x) \geq n\}$ $is\ measurable.$

$(ii) \Rightarrow (v) \quad (for\ \alpha = -\infty)$

By (iv), $\{x \,/ f(x) \leq n\}$ $is\ measurable.$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x \,/ f(x) \leq n\}\, is\ measurable.$

Hence $(ii), (iv) \Rightarrow (v)$.

Definition:

An extended real valued function f is said to be **Lebesgue measurable**, if its domain is measurable and it satisfies one of the first four statements of the above proposition.

Note:

$i)$ $\{x \,/ f(x) < \alpha\} = \{x \,/\, x \in f^{-1}(-\infty, \alpha)\}$

$ii)$ A continuous function on measurable set is measurable.

For any real α , $f^{-1}(-\infty, \alpha)$ is open.

$\Rightarrow f^{-1}(-\infty, \alpha)$ is measurable for all α.

Therefore, $\{x\ /f(x) < \alpha\}\ is\ measurable.\ \ \Rightarrow f\ is\ measurable.$

Recall a real valued function φ defined on an interval $[\,a, b]$ is called a **step function** if there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that for all i, the function φ assumes only one value in the interval (x_i, x_{i+1}) .

$iii)$ If f is a step function, then $f^{-1}\,(\alpha, \infty)$ is an interval (or) union of intervals.

$\Rightarrow\ f^{-1}\,(\alpha, \infty)$ is open for all real α. $\Rightarrow\ f^{-1}\,(\alpha, \infty)$ is measurable.

$\Rightarrow\ f$ is measurable.

$iv)$ If f is a measurable function and E is the measurable subset of the domain f. Then f/E is also measurable.

LEMMA:18

Let c be a constant and let $f\ and\ g$ be two measurable real valued functions defined on the same domain. Then the function $f + c$, cf , $f + g$, $f - g\ and\ fg$ are also measurable.

Proof:

$i)$ Let α be any real number.

Now, $\{x\ /f(x) + c < \alpha\} = \{x\ /\ f(x)\ < \alpha - c\}$

Since f is measurable, $\{x\ /\ f(x)\ < \alpha - c\}$ is measurable.

Therefore, Left hand side is measurable.

$\Rightarrow\ f + c$ is measurable.

$ii)$ <u>**Claim :**</u> $cf\ is\ measurable.$

If $c = 0$, then clearly $cf = 0\ \Rightarrow\ cf$ is measurable.

Suppose $c \neq 0$.

Then

$$\{x / cf < \alpha\} = \begin{cases} \{x / f(x) < \alpha/_c\}, if\ c > 0 \\ \\ \{x / f(x) > \alpha/_c\}, if\ c < 0 \end{cases}$$

Since f is measurable, R.H.S is measurable for all real α.

$\Rightarrow$ Left hand side is measurable for all real $\alpha \Rightarrow cf$ is measurable.

i$\boldsymbol{ii}$) **Claim** : $f + g\ is\ measurable$.

Suppose $f(x) + g(x) < \alpha$. Then $f(x) < \alpha - g(x)$.

There exist a rational number r such that $f(x) < r < \alpha - g(x)$.

Therefore $\{x / f(x) + g(x) < \alpha\} = \bigcup_r \{\{x/f(x) < r\} \cap \{x/g(x) < \alpha + r\}\}$

Since $f\ and\ g$ are measurable functions we have, $\{ x / f(x) < r\}$ and $\{ x / g(x) < \alpha + r\}$ are measurable. Therefore, Right hand side is measurable.

Hence $\{x / f(x) + g(x) < \alpha\}$ is measurable. $\Rightarrow f + g$ is measurable.

$\boldsymbol{iv}$) Since $-g = (-1) \times g$ is measurable by (ii)

Therefore, $f - g = f + (-g)$ is measurable.

v) Consider $\{x / f^2(x) > \alpha\} = \{x / f(x) > \sqrt{\alpha}\} \cup \{x / f(x) < -\sqrt{\alpha}\}$ for all $\alpha > 0$

Since f is measurable, we have $\{x / f^2(x) > \alpha\}$ is measurable, for all $\alpha > 0$.

Therefore, $\{x / f^2(x) > \alpha\} = D\ \ if\ \alpha < 0.$ $\Rightarrow f^2$ is measurable.

Now f , g are measurable $\Rightarrow f + g$ is measurable.

$\Rightarrow (f + g)^2$, f^2 , g^2 are measurable. $\Rightarrow (f + g)^2 - f^2 - g^2$ is measurable.

$\Rightarrow \frac{1}{2}[(f + g)^2 - f^2 - g^2]$ is measurable. $\Rightarrow fg$ is measurable.

LEMMA :19

Let $\{f_n\}$ be a sequence of measurable function (with the same domain Of definition) then the functions $\sup\ \{f_1, f_2, \dots, f_n\}$ *and* $\inf\ \{f_1, f_2, \dots, f_n\}$, $\sup_n f_n$, $\inf_n f_n$, $\overline{lim}\, f_n$ *and* $\underline{lim}\, f_n$ are all measurable.

Proof:

$i)$ Let $h(x) = \sup\ \{f_1(x), f_2(x), \dots, f_n(x)\}$

Now $\{x / h(x) > \alpha\} = \cup_{i=1}^{n} \{x / f_i(x) > \alpha\}$

Since each f_i is measurable, R.H.S is measurable.

Therefore, $\left\{x \frac{}{h(x)} > \alpha\right\}$ *is measurable*. Therefore, h is measurable.

$ii)$ Define $g(x) = \sup_n f_n(x)$.

Then $\{x / g(x) > \alpha\} = \cup_{i=1}^{\infty} \{x / f_i(x) > \alpha\}$

Since f_i is measurable, and $\mathfrak{m}$ *is a* $\sigma-$*algebra* . We have,

$\cup_{i=1}^{\infty} \{x / f_i(x) > \alpha\}$ *is measurable* $\Rightarrow$ $\{x / g(x) > \alpha\}$ is measurable.

$\Rightarrow$ g is measurable.

Similarly, we can prove $\inf\ \{f_1, f_2, \dots, f_n\}$ and $\inf_n f_n$ are measurable.

$iii)$ Since $\overline{lim}\, f_n = \inf_n \sup_{k \geq n} f_k$, Therefore, $\overline{lim}\ f_n$ is measurable.

Similarly, $\underline{lim}\, f_n$ is also measurable.

Definition:

A property is said to hold **almost everywhere**. If the set of points where it fails to hold is a set of measure zero.

LEMMA 20

If f is a measurable function, and $f = g$ is almost everywhere, then g is measurable.

Proof: Let $E = \{x / f(x) \neq g(x)\}$

Since $f = g$ is almost everywhere, we have $mE = 0$

Now ,

$\{x/g(x) > \alpha\} = \{x/f(x) > \alpha\} \cup \{x \in E/g(x) > \alpha\} \sim \{x \in E/g(x) \leq \alpha\}$

Since f is measurable, $\{x/f(x) > \alpha\}$ is measurable.

Since the second and third sets are subsets of E and $mE = 0$

$\Rightarrow$ Both the sets have measure zero and hence they are measurable.

Since $\mathfrak{m}$ *is a* $\sigma - algebra$, we have $\{x/g(x) > \alpha\}$ is measurable.

$\Rightarrow$ g is measurable.

LEMMA :21

Let f be a measurable function defined on [a,b] and assume that f takes the value $\pm \infty$ only on a set of measure zero. Then given $\varepsilon > 0$ we can find a step function g and a continuous function h such that $|f - g| < \varepsilon$ and $|f - h| < \varepsilon$

Proof:

To prove this we required following 4 lemmas.

Lemma:1

Given a measurable function f on [a , b] that takes the values $\pm \infty$ only on a set of measure zero and given $\epsilon > 0$ there is an integer M such that $|f| \leq M$ except on a set of measure less than $\varepsilon/3$.

Proof:

Suppose for all M such that $\{ x / |f(x)| > M\} \geq \varepsilon/3$.

Let $A_n = \{ x / |f(x)| > n\} \Rightarrow mA_n \geq \varepsilon/3 , for\ all\ n.$

Also, $A_1 \supset A_2 \supset A_3 \supset \cdots$ and let $A = \bigcap_{n=1}^{\infty} A_n$

$\Rightarrow$ $|f(x)| = \infty\ on\ A \Rightarrow mA = 0$

By Theorem 15, $0 = mA = \lim_{n\to\infty} mA_n \geq {}^{\varepsilon}/_{3}$. This is a contradiction.

Therefore given $\varepsilon > 0$, there is M such that $m\{\, x \,/\, \mid f(x) \mid > M\} < {}^{\varepsilon}/_{3}$.

Lemma:2

Let f be a measurable function on [a, b], given $\varepsilon > 0$ and M, there is a simple function φ $such\ that \mid f(x) - \varphi(x) \mid < \varepsilon$ except where $\mid f(x) \mid \geq M$. If $m \leq f \leq M$, then we may take φ so that $m \leq \varphi \leq M$.

Proof:

Given $\varepsilon > 0$, there exists M with $\mid f(x) \mid < M$ such that $\frac{M}{n} < \varepsilon$ for some n.

Let $E_k = \{x / \frac{(k-1)M}{n} < f(x) \leq \frac{kM}{n}\}$

Define $\varphi(x) = \frac{kM}{n},\ x \in E_k\ , -n \leq k \leq n$

Then on E_k , $\mid f(x) - \varphi(x) \mid \leq \frac{MK}{n} - \frac{M(K-1)}{n}$

$$= \frac{M}{n} < \varepsilon\ , for\ all\ k$$

Therefore, $\mid f(x) - \varphi(x) \mid < \varepsilon\ , for\ all\ x \in [a\ , b]$ except where $\mid f(x) \mid \geq M$.

If $-M \leq f(x) \leq M,\ (ie) \mid f(x) \mid \leq M$.

Then by the above construction, there exists a simple function $\varphi\ such\ that\ -M \leq f(x) \leq M, (ie) \mid f(x) \mid \leq M\ with\ \mid f(x) - \varphi(x) \mid < \varepsilon$.

Hence the lemma.

Lemma :3

Given a simple function $on\ [a\ , b]$, there is a step function g on [a , b] such that $g(x) = \varphi(x)$ except on a set of measure less than ${}^{\varepsilon}/_{2}$.

Proof:

Let φ be a simple function and it assumes finite number of values $C_1\ , C_2\ , C_3\ , \dots , C_n$. Let $E_i = \{x/\varphi(x) = C_i\}$.

Then by, theorem 16 (vi)

Since E_i is measurable, (by definition of simple function) for every E_i there is a finite union of intervals $U_{1i}, U_{2i}, \ldots, U_{ni}$ and

$V_i = U_{1i} \cup U_{2i} \cup \ldots\ldots \cup U_{ni}$ such that $m^*(E_i \Delta V_i) < \frac{\epsilon}{2^n}$, $1 \le i \le n$.

Define $g(x) = c_i, \forall x \in V_i$

If $x \in E_i \cap V_i$, then $g(x) = \varphi(x)$. If $g(x) \neq \varphi(x)$, then $x \in E_i \Delta V_i$, for some i

Therefore, $\{ x / g(x) \neq \varphi(x)\} \subset \bigcup_{i=1}^{n} (Ei \Delta Vi)\}$

$$m^*\{ x / g(x) \neq \varphi(x) \} \le \sum_{i=1}^{n} m*(Ei \Delta Vi)$$

$$< \sum_{i=1}^{n} \frac{\epsilon}{2^n} = \frac{\epsilon}{2}$$

Therefore, $g(x) = \varphi(x)$, except on a set of measure less than $\frac{\varepsilon}{2}$.
If $m \le \varphi \le M$ then we take g so that $m \le g \le M$.

Definition:

The function χ_E is defined by $\chi_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \notin E \end{cases}$ is called the characteristic function of E.

A linear combination $\varphi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$ is called a simple function if the sets E_i are measurable.

Lemma :4

Given a step function g on [a,b] there is a continuous function h such that $g(x) = h(x)$ except on a set of measurable $< \frac{\epsilon}{3}$. If $m \le g \le M$, then we may take h so that $m \le h \le M$.

Proof:

Given a step function g on [a,b] such that $g(x) = c_i$, $x \in [x_{i-1}, x_i]$ for some subdivision of [a , b] and $a = x_0 \le x_1 \le x_2 \le \ldots\ldots \le x_n = b$.

Define

$$h(x) = \begin{cases} c_i & if\ x \in \left[x_{i-1}, x_i - \frac{\epsilon}{3(n-1)}\right],\ 1 \le i \le n \\ c_n & if\ x \in [x_{n-1}, x_n] \\ (1-\lambda)\left(x_i - \frac{\epsilon}{3(n-1)}\right) + \lambda x_i & if\ x \in [x_i - \frac{\epsilon}{3(n-1)}, x_i],\ 0 \le \lambda \le 1 \end{cases}$$

Then h(x) = g(x), if x $\notin \bigcup_{i=1}^{n-1}([x_i - \frac{\epsilon}{3(n-1)}, x_i])$

Now, m $(\bigcup_{i=1}^{n-1}([x_i - \frac{\epsilon}{3(n-1)}, x_i]) \le \sum_{i=1}^{n-1} m\,([x_i - \frac{\epsilon}{3(n-1)}, x_i])$

$$\le \sum_{i=1}^{n-1} \frac{\epsilon}{3(n-1)}$$

$$= \frac{\epsilon}{3}$$

Therefore, h(x) = g(x) except on a set of measure less than $\frac{\epsilon}{3}$.

Main proof:

Since f takes the value $\pm\infty$ only on a set of measure zero, we may assume that m ≤ f ≤ M.

Given ∈>0, Let f be a measurable function. Then by lemma 2, there exists a simple function φ with m ≤ φ ≤ M such that $|f(x) - \varphi(x)| < \frac{\epsilon}{4}$.

By lemma 3, there exists a step function g with m ≤ g ≤ M on [a,b] such that $|g(x) - \varphi(x)| < \frac{\epsilon}{4}$.

$$\Rightarrow |f(x) - g(x)| < |f(x) - \varphi(x)| + |\varphi(x) - g(x)|$$

$$< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.$$

By lemma 4, there exists a continuous function h with m ≤ h ≤ M such that $|g(x) - h(x)| < \frac{\epsilon}{2}$.

$$\Rightarrow |f(x) - h(x)| \le |f(x) - g(x)| + |g(x) - h(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Observation:

χ_A is measurable iff A is measurable .

Proof:

Let α be any real number .

$$\{x / \chi_A(x) > \alpha\} = \begin{cases} \varphi & if\ \alpha \geq 1 \\ A & if\ 0 \leq \alpha < 1 \\ R & if\ \alpha < 0 \end{cases} \quad \text{............(1)}$$

$\Rightarrow \{x / \chi_A(x) > \alpha\}$ is measurable for all α.

Conversely, χ_A is measurable. $\Rightarrow \{x / \chi_A(x) > \alpha\}$ is measurable.

$\Rightarrow$ A is measurable. (by (1))

1.5 : Little Wood's 3 principle

i) Every (measurable) set is nearly a finite union of intervals.

ii) Every (measurable) function is nearly continuous.

iii) Every (measurable) convergence sequence of function is nearly uniformly convergence.

The following proposition gives one version of the third principle.

PROPOSITION :22

Let E be a measurable set of finite measure and $\{f_n\}$ be a sequence of measurable functions defined on E. Let f be a real valued function such that for each x in E we have $f_n(x)$ converges to f(x). Then given $\in > 0$ and $\delta > 0$, there is a measurable set A subset of E with $m(A) < \delta$ and integer N such that $\forall\ x \notin A$ and $\forall\ n \geq N$, $|f_n(x) - f(x)| < \in$.

Proof:

Let $\in > 0$ be given. Let $G_n = \{x \in E / |f_n(x) - f(x)| \geq \in\}$

Let $E_N = \bigcup_{n=N}^{\infty} G_n = \{x \in E / |f_n(x) - f(x)| \geq \in$ for some $n \geq N\}$

Since f_n is measurable and f_n converges to f point wise.

We have, f is measurable.

Since f_n , f are measurable we have, G_n and hence E_N are measurable.

Also we have $E_{N+1} \subseteq E_N$

Since $E_1 \subset E$ and $m(E) < \infty$ we have $m(E_1) < \infty$.

Then by Theorem 15, we have $m(\bigcap_{N=1}^{\infty} E_N) = \lim_{N\to\infty} mE_N$.

By the definition of E_N, for all $x \in E$, there exists N such that $x \notin E_N$ as $f_n(x) \to f(x)$. *Thus* $\bigcap_{N=1}^{\infty} E_N = \varphi$.

$\Rightarrow 0 = m\varphi = \lim_{N\to\infty} mE_N$.

$\Rightarrow$ *Given* $\delta > 0$, there exist N such that $mE_n < \delta$

$\Rightarrow \quad m\{x \in E / |f_n(x) - f(x)| \geq \varepsilon \ \ for\ n \geq N\} < \delta$

Take $A = E_N$ then $mA < \delta$ *and* A *is measurable*.

Now , $\quad A^c = \{x \in E \ / \ | f_n(x) - f(x)| < \varepsilon\ for\ all\ n \geq N\}$

$\qquad | f_n(x) - f(x)| < \varepsilon\ for\ all\ n \geq N\ and\ \forall\ x \notin A$.

PROPOSITION :23

Let E be a measurable set of finite measure and $\{f_n\}$ be a sequence of measurable functions that converge to a real valued function f almost everywhere on E . Then given $\varepsilon > 0$ there is a set A subset of E with $mA < \delta$ and an N such that for all $x \notin A$, and all $n \geq N$, $\ | f_n(x) - f(x)| < \varepsilon$

Proof:

Given $f_n \to f$ pointwise almost everywhere on E.

$\Rightarrow$ There exists $B \subset E$ *such that* $f_n(x) \to f(x)$ *as* $n \to \infty$ *for all* $x \in E - B = E_1$ & $mB = 0$ *and* E_1 *is measurable* .

By proposition 22, Given $\varepsilon > 0$ there exists $\delta > 0$ and a set $A \subset E_1$ *with* $mA < \delta$ and such that every $x \in E_1 - A$ *and* $n \geq N$, $| f_n(x) - f(x)| < \varepsilon$.

Let $A_1 = A \cup B$

$\Rightarrow mA_1 \leq m(A) + m(B) \ \Rightarrow mA_1 < \delta + 0 = \delta$

$\Rightarrow mA_1 < \delta$ and $| f_n(x) - f(x)| < \varepsilon\ for\ all\ n \geq N, x \notin A$.

UNIT-II

Lebesgue Integral

2.1 The Riemann integral

Let f be a bounded real valued function defined on the interval [a, b] and let $a = \xi_0 < \xi_1 < \cdots \xi_n = b$ be a subdivision of [a , b]. Then for each subdivision we can define the sums $S = \sum_{i=1}^{n} (\xi_i - \xi_{i-1})\, M_i$ and $s = \sum_{i=1}^{n} (\xi_i - \xi_{i-1})\, m_i$, where $M_i = \underset{[\varepsilon_{i-1}\,,\,\varepsilon_i]}{Sup} f(x)$ and

$m_i = \underset{[\varepsilon_{i-1}\,,\,\varepsilon_i]}{inf} f(x)$. Then we define upper Riemann integral of f by $R\int_a^{\bar{b}} f(x)dx = \inf S$, where the inf is taken over all possible sub divisions of [a,b].

Similarly, we can define lower Riemann integral and $R\int_{\underline{a}}^{b} f(x)dx = \sup s$

The upper integral is always at least as large as the lower integral and if the two are equal, we say that f is Riemann integrable and call this common value the Riemann integrable of f and we shall denote it by $R\int_a^b f(x)dx$.

By a step function we mean a function Ψ which has the form $\Psi(x) = C_i$, $\xi_{i-1} \le x \le \xi_i$ for some subdivision of [a ,b] and for some set of constant C_i . Under this definition we have $\int_a^b \psi(x)dx = \sum_{i=1}^{n} C_i(\xi_i - \xi_{i-1})$ with this definition we have, $R\int_a^{\bar{b}} f(x)dx = \inf \int_a^b \Psi(x)\, dx$ for all step functions $\Psi(x) \ge f(x)$.

Similarly, $R\int_{\underline{a}}^{b} f(x)dx = \underset{\varphi(x) \le f(x)}{\sup} \int_a^b \varphi(x)\, dx$

Example:

Show that if $f(x) = \begin{cases} 0 & \text{If x is irrational} \\ 1 & \text{if x is rational} \end{cases}$

Then f is not Riemann integrable.

Soln:

Let $a = \xi_0 < \xi_1 < \cdots \xi_n = b$

$m_i = \inf\ f(x) = 0$ and $M_i = \sup\ f(x) = 1,\ on\ [\xi_{i-1}, \xi_i]$.

This is true for every subdivision of [a , b].

$$S = \sum_{i=1}^{n}(\xi_i - \xi_{i-1})M_i = \sum_{i=1}^{n}(\xi_i - \xi_{i-1}) \quad [since\ M_i = 1]$$

$$= (\xi_1 - \xi_0) + (\xi_2 - \xi_1) + \cdots + (\xi_n - \xi_{n-1})$$

$$= \xi_n - \xi_0$$

$$= b - a \qquad \text{and } s = \sum_{i=1}^{n}(\xi_i - \xi_{i-1})\, m_i = 0$$

$$R\int_a^{\bar{b}} f(x)dx = \inf S = b - a \text{ and } R\int_{\underline{a}}^{b} f(x)dx = \sup s = 0$$

$$\Rightarrow R\int_a^{\bar{b}} f(x)dx \neq R\int_{\underline{a}}^{b} f(x)dx \Rightarrow f \text{ is not Riemann integrable.}$$

2.2: The Lebesgue Integral of a bounded function over a set of finite measure.

The function χ_E defined by,

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

is called the **characteristic function** of E.

A linear combination $\varphi(x) = \sum_{n=1}^{n} a_i\, \chi_{E_i}(x)$ is called a **simple function** if the sets E_i are measurable .

This representation for φ is not unique. However note that the function φ is simple iff it measurable and assumes only finite numbers of values.

If φ is a simple function and $\{a_1, a_2, \dots\dots, a_n\}$ the set of non-zero values of φ, then $\varphi = \sum a_i \chi_{A_i}$ where $A_i = \{x/\varphi(x) = a_i\}$, this representation of φ is called the **Canonical representation**.

If φ vanishes outside a set of finite measure, we define the integral of φ by, $\int \varphi(x)\, dx = \sum_{i=1}^{n} a_i\, mA_i$ where φ has the canonical representation

$\varphi = \int \sum a_i\, \chi_{A_i}$.

Note:

Some times we denote it by $\int \varphi$. If E is any measurable set then we define $\int_E \varphi = \int \varphi\, \chi_E$

LEMMA :1

Let $\varphi = \sum_{i=1}^{n} a_i\, \chi_{E_i}$ with $E_i \cap E_j = \emptyset$ for $\neq j$.

Suppose each set E_i is a measurable set of finite measure , then

$\int \varphi = \sum_{i=1}^{n} a_i\, mE_i$.

Proof:

Consider $A_a = \{x/\varphi(x) = a\}$

$$= \bigcup_{a_i = a} E_i$$

$$mA_a = \sum_{a_i = a} mE_i$$

$$\Rightarrow a\, mA_a = a \sum_{a_i = a} mE_i$$

$$\Rightarrow a\, mA_a = \sum_{a_i = a} a_i\, mE_i$$

$$\int \varphi\,(x) dx = \sum a_i\, mE_i$$

PROPOSITION:2

Let Ψ and φ be two simple functions which vanish out side a set of finite measure then $\int a\varphi + b\,\Psi =$ a $\int \varphi + b \int \Psi$ and if $\varphi \geq \Psi$ almost everywhere, then $\int \varphi \geq \int \Psi$.

Proof:

Let $\{A_i\}$ and $\{B_i\}$ be the sets which occur in the canonical representation of φ *and* Ψ

Let A_0 and B_0 be the sets where $\varphi\ and\ \Psi$ are zero. Then the set E_k obtained by taking all the intersections $A_i \cap B_j$ from a finite disjoint Collection of measurable sets.

And we may write $\varphi = \sum_{i=1}^{N} a_i\ \chi_{E_i}$ and $\Psi = \sum_{i=1}^{N} b_i\ \chi_{E_i}$.

Then by lemma 1, $\int \varphi = \sum_{i=1}^{N} a_i mE_i$ and $\int \Psi = \sum_{i=1}^{N} b_i\ mE_i$

Now $a\varphi + b\Psi = \sum_{i=1}^{N} (aa_i + bb_k)\chi_{E_i}$

Then by lemma 1 , $\int\ a\varphi + b\Psi = \sum_{i=1}^{N} (aa_i + bb_i) mE_i$

$$\int\ a\varphi + b\Psi = \sum_{i=1}^{N} aa_i\, mE_i + \sum_{i=1}^{N}\ bb_i mE_i$$

$$= a\sum_{i=1}^{N} a_i\, mE_i + b\sum_{i=1}^{N}\ b_i mE_i\,.$$

$$= \mathrm{a}\int \varphi + b\int \varphi$$

Note that $\int \varphi - \int \Psi = \int (\varphi - \Psi)$

$If\ \ \varphi \geq\ \Psi$ almost everywhere, then $\varphi - \Psi \geq 0$ almost everywhere.

$\Rightarrow \varphi - \Psi \geq 0$ on F and $mF^c = 0$

Now $E_k \cap F =\ E_k \cap F^c$ from a disjoint collection of measurable sets.

Now by definition,

$$\int (\varphi - \Psi) = \sum_{i=1}^{N} (a_i - b_i\ )\, \chi_{E_i}$$

$$= \sum_{i=1}^{N} c_i\, \chi_{\mathrm{E_i}} \text{ where } c_i = a_i - b_i,\ i = 1, 2, \dots, N$$

Now $E_i = E_i \cap (F \cup F^c) = (E_i \cap F) \cup (E_i \cap F^c)$

And $(E_i \cap F)\ and\ (E_i \cap F^c)$ are disjoint.

Since m is additive, $mE_i = m(E_i \cap F) + m(E_i \cap F^c)$

Now $\int \varphi - \int \Psi = \int (\varphi - \Psi)$

$$= \sum_{i=1}^{N} c_i\, mE_i = \sum_{i=1}^{N} c_i\ [m(E_i \cap F) + m(E_i \cap F^c)]$$

$$= \sum_{i=1}^{N} c_i\, m(E_i \cap F) + \sum_{i=1}^{N} c_i m(E_i \cap F^c)$$

$$= \sum_{i=1}^{N} c_i\, m(E_i \cap F) \text{ (as } E_i \cap F^c \subset F^c\ and\ mF^c = 0 \text{)}$$

$$\int \varphi - \int \Psi \geq 0 \text{ (as } \varphi \geq \Psi,\ c_i = a_i - b_i\ on\ E_i)$$

$$\Rightarrow \int \varphi \geq \int \Psi$$

PROPOSITION:3

Let f be defined and bounded on a measurable sets E with $mE < \infty$. In order that, $\underset{f \leq \psi}{inf} \int_E \psi(x)dx = \underset{f \geq \varphi}{sup} \int_E \varphi(x)dx$ for all simple function φ and ψ it is necessary and sufficient that f be measurable.

Proof:

Let f be bounded by M. Suppose f is measurable.

Let $E_k = \left\{x / \frac{kM}{n} \geq f(x) > \frac{(k-1)M}{n}\right\}$, $\quad -n \leq k \leq n$

Then E_k are measurable and disjoint.

$E = \bigcup_{k=-n}^{n} E_k \quad \Rightarrow \quad mE = \sum_{k=-n}^{n} mE_k$ -------------------(1)

Define the simple function

$$\psi(x) = \frac{M}{n} \sum_{k=-n}^{n} k\, \chi_{E_k}(x)$$

$$\varphi(x) = \frac{M}{n} \sum_{k=-n}^{n} (k-1)\; \chi_{E_k}(x)$$

$\Rightarrow \varphi_n(x) \leq f(x) \leq \psi_n(x)\ forall\ x \in E$

$\Rightarrow \underset{\varphi \leq f}{sup} \int \varphi\, dx \geq \int \varphi_n\, dx$

$= \frac{M}{n} \sum_{k=-n}^{n} (k-1) m\, E_k$ -------------(2)

Similarly, $\underset{\psi \geq f}{inf} \int \psi\, dx \leq \int \psi_n\, dx$ = $\frac{M}{n} \sum_{k=-n}^{n} k\, mE_k$ --------------------(3)

From (1) and (2) ,

$$\Rightarrow \quad 0 \leq \underset{\psi \geq f}{inf} \int \psi\, dx - \underset{\varphi \leq f}{sup} \int \varphi\, dx$$

$$\leq \frac{M}{n} \sum_{k=-n}^{n} k\, mE_k - \frac{M}{n} \sum_{k=-n}^{n} (k-1) mE_k$$

$$= \frac{M}{n} \sum_{k=-n}^{n} m\, E_k$$

$$= \frac{M}{n}\ mE \quad [by\ (1)]$$

Since n is arbitrary and mE < ∞, we have

$$\Rightarrow \inf_{\psi \geq f} \int_E \psi dx = \sup_{f \geq \varphi} \int_E \varphi dx$$

Conversely,

Suppose $\inf_{\psi \geq f} \int_E \psi\, dx = \sup_{f \geq \varphi} \int_E \varphi\, dx$

Given n, there exist a simple functions $\varphi_n\ and\ \psi_n$ such that

(i) $\varphi_n(x) \leq f(x) \leq \psi_n(x)$

(ii) $\int \psi_n(\mathrm{x}) - \int \varphi_n(x) < \frac{1}{\mathrm{n}}$

Define $\psi^* = inf\ \psi_n$

$\varphi^* = \sup\ \varphi_n$

Then $\varphi^*\ and\ \psi^*$ are measurable.

Also, $\varphi_n(x) \leq f(x) \leq \psi_n(x)\ \forall\ n$

$\Rightarrow \sup\ \varphi_n(x) \leq f(x) \leq \inf\ \psi_n(x)$

$\Rightarrow \varphi^*(x) \leq f(x) \leq \psi^*(x)$

Let $\Delta = \{x/\varphi^*(x) < \psi^*(x)\}$

$\Delta_\gamma = \{x/\varphi^*(x) < \psi^*(x)\} - \frac{1}{\gamma}$

$Cleary$, $\Delta_\gamma \subset \{x/\varphi_n(x) < \psi_n(x)\} - \frac{1}{\gamma}$

Now Claim that $m\{\, x/\varphi_n(x) < \psi_n(x)\} - \frac{1}{\gamma}\} < \frac{\gamma}{n}$

Let $R_n = \{x/\varphi_n(x) < \psi_n(x)\} - \frac{1}{\gamma}\}$

$\Rightarrow \psi_n(x) - \varphi_n(x) > \frac{1}{\gamma}$ on R_n

$\Rightarrow \psi_n(x) - \varphi_n(x) > \frac{1}{\gamma}\ \chi_{R_n}(x)$

$\Rightarrow \int(\psi_n - \varphi_n) > \frac{1}{\gamma}\ m\ (R_n) \quad [\int \chi_{R_n} = m\ (R_n)]$

$\Rightarrow \int \psi_n - \int \varphi_n > \frac{1}{\gamma}\, m\,(R_n)$

$\Rightarrow \quad \frac{1}{n} > \int \psi_n - \int \varphi_n > \frac{1}{\gamma}\, m\,(R_n)$

$\Rightarrow \quad m\,(R_n) < \frac{\gamma}{n}$

Since n is arbitrary, $m\,(R_n) = 0 \quad \Rightarrow \quad m\left(\Delta_\gamma\right) = 0$

As $\Delta = \bigcup_{\gamma=1}^{\infty} \Delta_\gamma \Rightarrow m\Delta = 0$

$\Rightarrow \quad \varphi^* = \psi^*$ except on a set of measure zero.

$\Rightarrow \quad \varphi^* = \psi^*$ almost everywhere.

$\Rightarrow \quad \varphi^* = f$ almost everywhere on E, $\varphi^* \leq f \leq \psi^*$

$\Rightarrow \quad$ f is measurable.

Definition:

If f is a bounded measurable function defined on a measurable set E with $mE < \infty$. We define the (lebesgue) integral of f over E by,

$$\int_E f(x)dx = inf \int_E \Psi(x)dx, for\ all\ simple\ \ function\ \Psi \geq f$$

Note:

(i) We write the integral as $\int_E f$.

(ii) If $E = [a, b]$ we write $\int_a^b f$ instead of $\int_{[a,b]} f$.

(iii) If f is a bounded measurable function which vanishes outside a set E of finite measure, we write $\int f$ for $\int_E f$.

(iv) $\int_E f$ is the same as $\int f\, \boldsymbol{\chi_E}$.

PROPOSITION :4

Let f be a bounded function defined on [a , b]. If f is Riemann integrable on [a ,b], then it is measurable *and* $R\int_a^b f(x)dx = \int_a^b f(x)dx$.

Proof:

Since the step function is also a simple function, we have

$$R\int_{\underline{a}}^{b} f(x)dx \le \sup_{\varphi \le f}\int_a^b \varphi(x)dx \le \inf_{f \le \psi}\int_a^b \psi(x)dx$$

$$\le R\int_a^{\overline{b}} f(x)dx$$

Since f is Riemann integerable we have, $R\int_{\underline{a}}^{b} f(x)dx = R\int_a^{\overline{b}} f(x)dx$

$\Rightarrow \quad \sup_{\varphi \le f}\int_a^b \varphi(x)dx = \inf_{f \le \psi}\int_a^b \psi(x)dx$

$\Rightarrow$ f is measurable (By proposition 3)

Also from the above relation we have, $R\int_a^b f(x)dx = \inf_{\psi \ge f}\int_a^b \psi(x)dx$.

Therefore, $R\int_a^b f(x)dx = \int_a^b f(x)dx$.

PROPOSITION:5

If f *and* g are bounded measurable functions defined on a set E of finite measure, then

i. $\int_E af + bg$ = a$\int_E f$ + b$\int_E g$
ii. If f = g almost everywhere then $\int_E f = \int_E g$
iii. If f ≤ g almost everywhere then $\int_E f \le \int_E g$ and hence $|\int f| \le \int |f|$
iv. If A ≤ f(x) ≤ B then AmE ≤ $\int f$ ≤ BmE
v. If A & B are disjoint measurable sets of finite measure then $\int_{A\cup B} f = \int_A f + \int_B f$

Proof: Suppose a > 0

$\int_E af = \inf_{\Psi \ge f}\int_E a\Psi$ [∵ Ψ ≥ f ⟺ aΨ ≥ af]

$= a\inf_{\Psi \ge f}\int_E \Psi$

$\int_E af = a\int_E f \quad \longrightarrow (1)$

Suppose a < 0

$\int_E af = \inf_{\phi \le f}\int_E a\phi$ [∵ ϕ ≤ f ⟺ aϕ ≥ af , (a<0)]

$= a \sup_{\phi \le f} \int_E \phi$

$= a \inf_{\Psi \ge f} \int_E \Psi$ (by proposition 3)

$\int_E af = a\int_E f \longrightarrow (2)$

From (1) & (2), $\int_E af = a\int_E f \longrightarrow (3)$

If Ψ_1 and Ψ_2 are simple functions such that $\Psi_1 \ge$ f and $\Psi_2 \ge$ g.

Then $\Psi_1 + \Psi_2$ is a simple function and $\Psi_1 + \Psi_2 \ge$ f+g

$\therefore \int_E f + g \le \int_E \Psi_1 + \Psi_2$

$= \int_E \Psi_1 + \int_E \Psi_2$

Now by taking infimum on R.H.S over $\Psi_1 \ge$ f and $\Psi_2 \ge$ g.

Then we have $\int_E f + g \le \inf_{\Psi_1 \ge f} \int_E \Psi_1 + \inf_{\Psi_2 \ge g} \int_E \Psi_2$

$= \int_E f + \int_E g \longrightarrow (4)$

On the other hand, if ϕ_1 and ϕ_2 are simple functions such that

$\phi_1 \le$ f and $\phi_2 \le$ g.

Then $\phi_1 + \phi_2$ is a simple function and $\phi_1 + \phi_2 \le$ f+g

$\therefore \int_E f + g \ge \int_E \phi_1 + \phi_2 = \int_E \phi_1 + \int_E \phi_2$

Now by taking sup on R.H.S over $\phi_1 \le$ f and $\phi_2 \le$ g

Then $\int_E f + g \ge \sup_{\phi_1 \le f} \int \phi_1 + \sup_{\phi_2 \le g} \int \phi_2$

$= \int_E f + \int_E g \longrightarrow (5)$

From (4) & (5), we have $\int_E f + g = \int_E f + \int_E g \longrightarrow (6)$

i) $\int_E af + bg = \int_E af + \int_E bg$ (by (6))

$= a \int_E f + b \int_E g$ (by (3))

ii) Given f = g almost everywhere

$\Rightarrow$ f - g = 0 almost everywhere

$\Psi \ge$ f - g $\Rightarrow \Psi \ge 0$ almost everywhere

$\Rightarrow \int_E \Psi \geq 0$ (by proposition 2)

Taking infimum we have, $\inf_{\Psi \geq f-g} \int_E \Psi \geq 0$

$\Rightarrow \int_E f - g \geq 0$

Similarly, we can prove $\int_E f - g \leq 0$

$\therefore \int_E f - g = 0$

$\Rightarrow \int_E f - \int_E g = 0$ (by (i))

$\Rightarrow \int_E f = \int_E g$

iii) Suppose f ≤ g almost everywhere

$\Rightarrow$ f - g ≤ 0 almost everywhere

$\phi \leq f - g \Rightarrow \phi \leq 0$ almost everywhere $\Rightarrow \int_E \phi \leq 0$

$\Rightarrow \sup_{\phi \leq f-g} \int_E \phi \leq 0$

$\Rightarrow \int_E f - g \leq 0 \Rightarrow \int_E f - \int_E g \leq 0 \Rightarrow \int_E f \leq \int_E g$

Since $f \leq |f|$ and $-f \leq |f|$

$\Rightarrow \int_E f \leq \int_E |f|$ and $\int_E -f \leq \int_E |f|$

$\Rightarrow \int_E f \leq \int_E |f|$ and $-\int_E f \leq \int_E |f|$

$\Rightarrow \left|\int_E f\right| \leq \int_E |f|$

iv) Suppose $A \leq f(x) \leq B$

$\Rightarrow \int_E A \leq \int_E f \leq \int_E B$

$\Rightarrow A\, mE \leq \int_E f \leq B\, mE$ $[\because \int_E A = A\int_E 1 = A\, mE]$

v) Suppose A & B are disjoint measurable sets of finite measure

Now, $\chi_{A\cup B}(x) = \begin{cases} 1 \; if \; x \in A \cup B \\ 0 \; if \; x \notin A \cup B \end{cases}$

Since A & B are disjoint measurable sets, then we have

$\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x)$

$\int_{A\cup B} f = \int f\,\chi_{A\cup B} = \int f(\chi_A + \chi_B) = \int (f\chi_A + f\chi_B) = \int f\chi_A + \int f\chi_B$

Therefore, $\int_{A\cup B} f = \int_A f + \int_B f$.

PROPOSITON:6 [Bounded Convergence Theorem]

Let $\{f_n\}$ be a seqence of measurable functions defined on a set E of finite measure and Suppose that there is a real number M such that $|f_n(x)| \leq$ M for all n, for all x and f(x) = $\lim_{n\to\infty} f_n(x)$ for all x∈E then $\int_E f = \lim_{n\to\infty} f_n$.

Proof :

Given $\varepsilon > 0$, there exists N $>$ 0 and a measurable set A ⊂ B with mA$< \frac{\varepsilon}{4M}$ such that for all n ≥ N and x ∈ E - A, $|f_n(x) - f(x)| < \frac{\varepsilon}{2mE}$ ⟶(1) (by proposition 22 of unit I) .

Now, $\left|\int_E f_n - \int_E f\right| = \left|\int_E f_n - f\right|$

$$\leq \int_E |f_n - f|$$

$$= \int_A |f_n - f| + \int_{E-A} |f_n - f|$$

$$\leq \int_A 2M + \int_{E-A} \frac{\varepsilon}{2mE}$$

$$= 2M\,mA + \frac{\varepsilon}{2mE}\,m(E-A) < 2M\,\frac{\varepsilon}{4M} + \frac{\varepsilon}{2} = \boldsymbol{\varepsilon}$$

$\therefore \left|\int_E f_n - \int_E f\right| < \boldsymbol{\varepsilon}$ for all n ≥ N. Therefore, $\int_E f = \lim_{n\to\infty} \int_E f_n$

Definition:

Let f be a non-negative measurable function defined on a measurable set E, we define $\int_E f = \sup_{h\leq f} \int_E h$, where h is a bounded measurable function such that m$\{x/h(x) \neq 0\}$ is finite.

[(ie) h vanishes outside a set of finite measure]

PROPOSITION:7

A bounded function f on [a,b] is Riemann integrable iff the set of points at which f is discontinuous has measure zero.

PROPOSITION:8

If f and g are non-negative measurable functions then

i. $\int_E cf$ = c$\int_E f$, c >0
ii. $\int_E f + g$ = $\int_E f$ + $\int_E g$
iii. If f ≤ g almost everywhere , then $\int_E f \leq \int_E g$

Proof: i) Let f be a non-negative measurable function and c >0. For every bounded measurable function h, h ≤ f ⇒ ch ≤ cf and h_1 ≤ cf ⇒ $\frac{h_1}{c}$ ≤ f.

Now $\int_E cf$ = $\sup_{h_1 \leq f} \int_E h_1$ = $\sup_{\frac{h_1}{c} \leq f} c \int_E \frac{h_1}{c}$

= $\sup_{h \leq f} c \int_E h$ = c $\sup_{h \leq f} \int_E h$ = c$\int_E f$

ii) Let h and k be bounded measurable functions vanishing outside the set of finite measure.

h ≤ f, k ≤ g ⇒ h+k ≤ f+g ⇒ $\int_E h + k \leq \int_E f + g$

⇒ $\int_E h$ + $\int_E k \leq \int_E f + g$

Taking supremum on L.H.S over h ≤ f and k ≤ g then

⇒ $\sup_{h \leq f} \int_E h$ + $\sup_{k \leq g} \int_E k \leq \int_E f + g$

⇒ $\int_E f$ + $\int_E g \leq \int_E f + g$ ⟶(1)

Let l be a bounded measurable functions which vanishes outside a set of finite measure and l ≤ f + g. Define h = min(f, l) and k(x) = $l(x)$ - h(x)

Therefore, k = l - h is defined at all points of its domain. (since l is bounded, h is bounded)

By definition, h(x) ≤ f(x) and also, $l(x)$ ≤ f(x) + g(x)

⇒ h(x) + k(x) = $l(x)$ ≤ f(x) + g(x)

Therefore, 0 ≤ k(x) ≤ g(x) [as h = min (f, l)]

Moreover, h, k ≤ l ⇒ h, k are bounded measurable functions and they vanish outside the set of finite measure.

h ≤ f and k ≤ g

$\Rightarrow \int_E l = \int_E (k + h)$

$\Rightarrow \int_E l = \int_E k + \int_E h$

$\Rightarrow \int_E l \leq \int_E g + \int_E f$

Taking supremum on $l \leq$ f + g, $\Rightarrow \int_E f + g \leq \int_E f + \int_E g \quad \longrightarrow(2)$

From (1) &(2)

$$\int_E f + g = \int_E f + \int_E g$$

iii) Suppose f ≤ g almost everywhere

Let h be a bounded measurable function which vanishes outside the set of finite measure and h ≤ f - g

$\Rightarrow$ h ≤ 0 almost everywhere [∵ f ≤ g a.e $\Rightarrow$ h = f - g ≤ 0 a.e]

$\Rightarrow \int_E h \leq 0$

By taking supremum we have, $\sup_{h \leq f-g} \int_E h \leq 0$

$\Rightarrow \int_E f - g \leq 0$

Assume $\int_E g < \infty$. [∵ suppose $\int_E g = \infty$, then $\int_E f \leq \int_E g$]

Adding $\int_E g$ on both sides,

$\Rightarrow \int_E f - g + \int_E g \leq \int_E g$

$\Rightarrow \int_E (f - g + g) \leq \int_E g$

$\Rightarrow \int_E f \leq \int_E g$

THEOREM:9. [Fatou's Lemma]

If $\{f_n\}$ is a sequence of non-negative measurable functions and $f_n(x) \rightarrow f(x)$ almost everywhere on a set E, then $\int_E f \leq \underline{\lim} \int_E f_n$.

Proof:

Without loss of generality we may assume that, the convergence is everywhere, since the integrals over the sets of measure zero are zero.

Let h be a bounded measurable function which is not greater than f and which vanishes outside a set E′ of finite measure.

Define a function $h_n = \min\{h(x), f_n(x)\}$

Then h_n is bounded by the bound for h and vanishes outside E′.

And also $h_n(x) \to h(x)$. Then by proposition 6 (Bounded convergence theorem),

we have $\int_E h = \int_{E'} h = \lim \int_{E'} h_n$

$$\leq \underline{\lim} \int_E f_n$$

Now taking sup, we have $\sup_{h \leq f} \int_E h \leq \underline{\lim} \int_E f_n \implies \int_E f \leq \underline{\lim} \int_E f_n$.

THEOREM : 10.[Monotone Convergence Theorem]

Let $\{f_n\}$ be an increasing sequence of non-negative measurable functions and let $f = \lim f_n$ almost everywhere then $\int_E f = \lim \int_E f_n$.

Proof:

By Fatou's Lemma we have, $\int_E f \leq \underline{\lim} \int_E f_n$. $\longrightarrow$ (1)

But for each m we have, $f_n \leq f$ and also $\int_E f_n \leq \int_E f$

$\Rightarrow \overline{\lim} \int_E f_n = \int_E f$. $\longrightarrow$ (2)

From (1) & (2), $\int_E f \leq \underline{\lim} \int_E f_n \leq \overline{\lim} \int_E f_n \leq \int_E f$

$\Rightarrow \underline{\lim} \int_E f_n = \overline{\lim} \int_E f_n = \int_E f$

$\Rightarrow \lim \int_E f_n = \int_E f$.

COROLLARY : 11

Let $\{u_n\}$ be a sequence of non-negative measurable functions and let $f = \sum_{n=1}^{\infty} u_n$. Then $\int f = \sum_{n=1}^{\infty} \int u_n$.

Proof:

$f = \sum_{n=1}^{\infty} u_n$. Let $S_n = \sum_{k=1}^{n} u_k$

Since, $u_k \geq 0 \; for \; all \; k, \{S_n\}$ is an increasing sequence of non-negative measurable functions and $S_n \to f$.

By Monotone Convergence theorem, $\int f \; = \; lim_{n\to\infty} \int S_n$

$$= \lim_{n\to\infty} \int \sum_{k=1}^{n} u_k$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int u_k = \sum_{k=1}^{\infty} \int u_k$$

PROPOSITION :12

Let f be a non-negative function and $\{E_i\}$ a disjoint sequence of measurable sets. Let $E = \bigcup E_i$. Then $\int_E f = \sum \int_{E_i} f$

Proof:

Let $u_i = f \, \chi_{E_i}$

Since, $\{E_i\}$ are disjoint sequence of measurable sets and $E=\bigcup_{i=1}^{\infty} E_i$,we have

$\chi_E = \sum_{i=1}^{\infty} \chi_{E_i} \;\; \Rightarrow \;\; f\chi_E = \sum_{i=1}^{\infty} f\chi_{E_i} \; = \; \sum_{i=1}^{\infty} u_i$

By corollary 11, $\int f\chi_E = \sum_{i=1}^{\infty} \int u_i$

$$\int f\chi_E = \sum_{i=1}^{\infty} \int f\chi_{E_i}$$

$$\int_E f \; = \; \sum_{i=1}^{\infty} \; \int_{E_i} f$$

Definition :

A non-negative measurable function f is called integrable over the measurable set E if $\int_E f < \infty$.

PROPOSITION:13.

Let f and g be two non-negative measurable functions. If f is integrable over E and g(x) < f(x) on E. Then g is also integrable on E and

$\int_E f - g = \int_E f - \int_E g$.

Proof:

$$\int_E f = \int_E f - g + g$$

$$= \int_E f - g + \int_E g \quad \longrightarrow (1) \quad [as \, f - g \geq 0]$$

Since f integrable, $\int_E f < \infty. \Rightarrow \int_E f - g + \int_E g < \infty \; \Rightarrow \int_E g < \infty$.

$\Rightarrow$ g is integrable.

$(1) \Rightarrow \int_E (f - g) = \int_E f - \int_E g \qquad [as\ \int_E g < \infty]$.

PROSITION:14

Let f be a non-negative function which is integrable over a set E. Then given $\varepsilon > 0$, there exists a $\delta > 0$ such that for every set A subset of E with mA < δ we have $\int_A f < \varepsilon$

Proof:

Case (i):

Suppose f is bounded. Let $\varepsilon > 0$ be given.

If $A \subset E$ such that mA< δ then $| \int_A f | \leq \int_A | f |$

$$\leq \int_A M = M \,.\, m(A) < M \,.\, \delta = \varepsilon$$

$$\Rightarrow | \int_A f | < \varepsilon$$

Case(ii):

Given f, Define f_n= min {f , n}

$\Rightarrow f_n \leq n \Rightarrow$ each f_n is bounded and $\lim_{n\to\infty} f_n(x) = f(x)$

Also $\{f_n\}$ is an increasing sequence of measurable functions.

∴By Monotone Convergence Theorem, we have $\int f = \lim_{n\to\infty} \int f_n$

Let $\varepsilon > 0$ be given, there exists N such that $\int f_N > \int f - \frac{\varepsilon}{2}$

Now, $\int (f - f_N) = \int f - \int f_N < \frac{\varepsilon}{2}$. Let $\delta = \frac{\varepsilon}{2N}$.

If mA < δ then $\int_A f = \int_A (f - f_N) + f_N$

$$= \int_A f - f_N + \int_A f_N$$

$$= \frac{\varepsilon}{2} + N\, mA \qquad [\because f_N = \min(f, N)]$$

$$\int_A f < \frac{\varepsilon}{2} + N\,\delta = \frac{\varepsilon}{2} + N\,\frac{\varepsilon}{2N} = \varepsilon$$

$$\Rightarrow \int_A f < \varepsilon$$

Theorem: 15

Let $\{f_n\}$ be a sequence of non-negative measurable functions which converges to f and suppose $f_n \leq f$, for all n then $\int f = \lim_{n\to\infty} \int f_n$

Proof:

Let $\lim_{n\to\infty} \int f_n = f$

By Fatou's lemma, $\int f \leq \underline{lim} \int f_n$-----------(1)

Since $f_n \leq f$ we have, $\int f_n \leq \int f$, for all n.

$\Rightarrow \overline{lim} \int f_n \leq \int f$ ------------(2)

From (1) and (2), $\overline{lim} \int f_n \leq \int f \leq \underline{lim} \int f_n$

But $\underline{lim} \int f_n \leq \overline{lim} \int f_n$

$\Rightarrow \underline{lim} \int f_n = \overline{lim} \int f_n = \int f$

$\Rightarrow lim \int f_n$ exists and $lim \int f_n = \int f$

Example :

The Monotone Convergence theorem need not hold for decreasing sequence of f_n.

Soln: Consider the function $f_n(x) = \chi_{[n,\infty]}(x)$

Then $\int f_n = \int \chi_{[n,\infty]}(x)$ = m $[n, \infty)$ = ∞, for all n.

Also f_n is decreasing to zero function and so f = 0 = $\lim_{n\to\infty} f_n(x)$

$\Rightarrow \int f = 0$. But $\int f \neq \lim \int f_n$.

2.3 General Lebesgue Integral

Definition:

By the positive part f^+ of a function f, we mean the function f^+= f v o (ie) $f^+(x)$= max {f(x), 0}

Similarly, we define the negative part f^- by f^-= - f v o

(ie) $f^-(x)$= max {-f(x), 0} or f^-x = - min {f(x), 0}

If f is measurable and so $f^+$and $f^-$are measurable.

We have $f = f^+ - f^-$ and | f |= $f^+ + f^-$.

Definition:

A measurable function f is said to be integrable over E if f^+ and f^- are both integrable over E and we define $\int f = \int f^+ - \int f^-$.

PROPOSITION:16

Let f and g be integrable over E, then

(i) the function cf is integrable over E and $\int_E cf = c\int_E f$

(ii) the function f+g is integrable over E and $\int_E f + g = \int_E f + \int_E g$

(iii) If f ≤ g is a.e , then $\int_E f \leq \int_E g$

(iv) If A and B are disjoint measurable sets contained in E, then $\int_{A \cup B} f = \int_A f + \int_B f$

Proof:

(i) Suppose c ≥ 0

Then, cf = c $(f^+ - f^-)$

$= cf^+ - cf^-$

and $cf^+ and\ cf^-$ are non-negative integrable functions.

⇒ cf is integrable, when c ≥ 0

$\therefore \int_E cf = \int_E cf^+ - cf^-$

$= \int_E cf^+ - \int_E cf^-$ [By Definition]

$= c\int_E f^+ - c\int_E f^-$

$$= c[\int_E f^+ - \int_E f^-]$$

$$\int_E cf \ = \ c\int_E f$$

Suppose c < 0, Let c = -d, d>0

Now, cf = (-d)f

cf = (-d) $(f^+ - f^-)$ = - d$f^+ + df^-$ = df^-- df^+

and df^+ and df^- are non-negative integrable functions over E.

⇒ cf is integrable, when c < 0

$$\int_E cf = \int_E df^- - df^+$$

$$= \int_E df^- - \int_E df^+ \text{ (by definition)}$$

$$= d\int_E f^- - d\int_E f^+$$

$$= d[\int_E f^- - \int_E f^+]$$

$$= -c[\int_E f^- - \int_E f^+]$$

$$= c[\int_E f^+ - \int_E f^-]$$

$$\int_E cf = c\int_E f$$

(ii) Let $f = f_1 - f_2$ where f_1 and f_2 are non-negative integrable functions.

$$\Rightarrow f^+ - f^- = f_1 - f_2$$

$$\Rightarrow f^+ + f_2 = f_1 + f^-$$

$$\Rightarrow \int f^+ + f_2 = \int f_1 + f^-$$

$$\int f^+ + \int f_2 = \int f_1 + \int f^- \text{ (by proposition 8)}$$

$$\Rightarrow \int f^+ - \int f^- = \int f_1 - \int f_2$$

$$\Rightarrow \int f = \int f_1 - \int f_2 \text{ --------------(1)}$$

Now Suppose f and g are integrable, and

$$f + g = f^+ - f^- + g^+ - g^-$$

$$= (f^+ + g^+) - (f^- + g^-)$$

Also $f^+ + g^+, f^- + g^-$ are non-negative integrable functons over E.

By (1), $\int_E f+g = \int_E (f^+ + g^+) - \int_E (f^- + g^-)$

$$= \int_E f^+ + \int_E g^+ - \int_E f^- - \int_E g^-$$

$$= \int_E f^+ - \int_E f^- + \int_E g^+ - \int_E g^-$$

$$\int_E f+g = \int_E f + \int_E g$$

(iii) Let A = { x/ g(x) =∞}, B = { x/ f(x) = ∞}

$\Rightarrow$ B $\subseteq A$ (as f $\leq$ g)

On E-A, g - f is well-defined, finite and g - f $\geq$ 0 almost everywhere [as f and g are integrable]

Also mA=0, $\int_E g = \int_A g + \int_{E-A} g$

$$= \int_{E-A} g = \int_{E-A} f + (g-f)$$

$$= \int_{E-A} f + \int_{E-A} g - f$$

By proposition 8, $\int_{E-A} g - f \geq 0$ [∵ g-f $\geq$ 0 a.e on E-A]

$\Rightarrow \int_E g \geq \int_{E-A} f = \int_{E-A} f + \int_A f$ [∵ mA=0]

$$= \int_E f$$

$$\Rightarrow \int_E g \geq \int_E f$$

iv) $\int_{A\cup B} f = \int f\,\chi_{A\cup B} = \int f(\chi_A + \chi_B) = \int f\,\chi_A + f\,\chi_B$

$$= \int f\chi_A + \int f\chi_B = \int_A f + \int_B f.$$

Dominated convergence theorem (or) Lebesgue convergence theorem

THEOREM:17

Let g be integrable over E and Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq$ g on E and for almost all x on E, we have f(x) = $\lim_{n\to\infty} f_n$(x). Then $\int_E f = \lim_{n\to\infty} \int_E f_n$.

Proof:

Since g is integrable , f_n is measurable and $|f_n| \leq g$. We have each f_n is integrable.

Therefore, m $\{x|\, f_n(x) = \infty\} = 0, \forall\, n.$

So ignoring set of measure zero, we can assume $|f_n(x)| < \infty$ for all x ∈ E.

Consider g $-f_n$, Now $f_n \leq |f_n| \leq g \;\; \forall n \;\; \Rightarrow$ g$-f_n \geq 0$

Also f = $\lim_{n\to\infty} f_n \Rightarrow \lim_{n\to\infty}(g - f_n) = g - f$

By Fatou's lemma , $\int g - f \leq \underline{lim} \int (g - f_n)$.

$$\Rightarrow \int g - \int f \leq \int g - \overline{lim} \int f_n$$

$$\Rightarrow \int f \geq \overline{lim} \int f_n \longrightarrow ① \qquad [\because \int g < \infty]$$

Also $|f_n| \leq g \Rightarrow -$ g $\leq f_n \leq g$

$$\Rightarrow \text{g} + f_n \geq 0 , \forall\, n \text{ and } \lim_{n\to\infty} g + f_n = g + \text{f}$$

By Fatou's lemma , $\int g + f \leq \underline{lim} \int (g + f_n)$

$$\Rightarrow \int g + \int f \leq \int g + \underline{lim} \int f_n$$

$$\Rightarrow \int f \leq \underline{lim} \int f_n \longrightarrow ② \qquad [\because \int g < \infty]$$

From ① and ② ,

$$\int f \leq \underline{lim} \int f_n \leq \overline{lim} \int f_n \leq \int f$$

$$\Rightarrow \underline{lim} \int f_n = \overline{lim} \int f_n = \int f.$$

$\Rightarrow$ lim $\int f_n$ exists and lim $\int f_n = \int f$.

PROPOSITION:18 [Generalization of Lebesgue convergenceTheorem]

Let $\{g_n\}$ be a sequence of integrable functions which converges almost everwhere to an integrable function g. Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g_n$ and $\{f_n\} \rightarrow$f almost everwhere if $\int g = \lim \int g_n$ then $\int f = \lim \int f_n$.

Proof:

Let $g_n \to g$ almost everwhere.

Let D = { x / $g_n(x)$ does not converges to g(x) }. Then m D = 0

$\Rightarrow \int_D g_n = 0$, $\forall n$ and $\int_D g = 0$

So we can assume that $g_n(x) \to g(x)$, for all x ∈ E.

Similarly, we can assume that $f_n(x) \to f(x)$, for all x ∈ E.

Since g_n is measurable, f_n is measurable and $|f_n| \leq g_n$.

⇒ each f_n is integrable as each g_n is integrable.

Consider $g_n - f_n$, $g_n - f_n \geq 0$, $\forall\, n$

and $\lim_{n\to\infty}(g_n - f_n)$ = g – f , ignoring a set of measure zero.

By Fatou's lemma , $\int g - f \leq \underline{lim} \int g_n - f_n$.

$$\Rightarrow \int g - \int f \leq \underline{lim} \int g_n - \overline{lim} \int f_n.$$

$$\Rightarrow \int g - \int f = \int g - \overline{lim} \int f_n \qquad [\because \lim \int g_n = \int g]$$

$$\Rightarrow \int f \geq \overline{lim} \int f_n \qquad [\because \int g < \infty]$$

Similarly, we can prove $\int f \leq \underline{lim} \int f_n$. $\therefore \int f = \lim \int f_n$.

PROPOSITION:19

A measurable $f_n \to f$ is integrable over E iff $|f|$ is integrable.

Proof:

Suppose f is integrable over E and f $= f^+ - f^-$

⇒ f^+ & f^- are integrable (by definition) over E.

$$\int f^+ < \infty \text{ and } \int f^- < \infty$$

Now , $|f| = f^+ - f^-$

$$\Rightarrow \int |f| = \int f^+ + \int f^- < \infty$$

$$\Rightarrow \int |f| < \infty \Rightarrow |f| \text{ is integrable.}$$

Conversely ,

Suppose $|f|$ is integrable over E.

To Prove: f is integrable.

Given f is measurable. $\Rightarrow$ f^+ are f^- are measurable

Also $|f| = f^+ + f^-$

Now $f^+ \leq |f|$ and $f^- \leq |f|$

$\Rightarrow$ $\int f^+ \leq \int |f| < \infty$ and $\int f^- \leq \int |f| < \infty$

$\Rightarrow$ f^+ and f^- are integrable $\Rightarrow$ f is integrable.

PROPOSITION:20

If f is integrable over E, then $|\int f| \leq \int |f|$.

Proof:

Since $f \leq |f|$ and $-f \leq |f|$

$\Rightarrow$ $\int f \leq \int |f|$ and $-\int f \leq \int |f|$

$\Rightarrow$ $\int f \geq -\int |f|$

$\Rightarrow -\int |f| \leq \int f \leq \int |f|$

$\Rightarrow$ $|\int f| \leq \int |f|$.

Example:

Prove that the function $\frac{\sin x}{x}$ is not lebesgue integrable over $[0, \infty)$.

Soln :

We know that the measurable function f is integrable iff $|f|$ is integrable.

Now, consider the integral,

$$\int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \left|\frac{\sin x}{x}\right| dx$$

$$= \sum_{r=1}^{n} \int_0^{\pi} \left|\frac{\sin(y+(r-1)\pi)}{(y+(r-1)\pi)}\right| dy$$

$[put\ x = (y + (r-1)\pi)\ , dx = dy$

$x = (r-1)\pi\ ,\ y = 0$

$$x = r\pi \ , y = \pi]$$

$$= \sum_{r=1}^{n} \int_0^{\pi} \left| \frac{\sin(y+(r-1)\pi)}{r\pi} \right| dy$$

$$[(y + (r-1)\pi \leq r\pi \, , \ 0 \leq y \leq \pi]$$

$$\geq \sum_{r=1}^{n} \frac{1}{r\pi} \int_0^{\pi} \left| \sin(y + (r-1)\pi) \right| dy$$

$$= \sum_{r=1}^{n} \frac{1}{r\pi} \int_0^{\pi} \left| \sin \ y \right| dy$$

$$= \sum_{r=1}^{n} \frac{1}{r\pi} \int_0^{\pi} \sin \ y \, dy$$

$$= \sum_{r=1}^{n} \frac{1}{r\pi} \left[-\cos y \right] \begin{matrix} \pi \\ 0 \end{matrix}$$

$$= \sum_{r=1}^{n} \frac{1}{r\pi} \left[-\cos \pi + \cos 0 \right] = \sum_{r=1}^{n} \frac{2}{r\pi} = \frac{2}{\pi} \sum_{r=1}^{n} \frac{1}{r}$$

$$= \lim_{n\to\infty} \int_0^{\infty} \left| \frac{\sin x}{x} \right| dx$$

$$\geq \lim_{n\to\infty} \frac{2}{\pi} \sum_{r=1}^{n} \frac{1}{r} = \frac{2}{\pi} \sum_{r=1}^{\infty} \frac{1}{r} = \infty$$

$$\int_0^{n\pi} \left| \frac{\sin x}{x} \right| dx = \infty.$$

UNIT III

COMPLEX ANALYSIS

3.1 Complex Numbers

Algebra of complex numbers:

Definition:

A complex number is an ordered pair of numbers C = {(a,b) / a,b ϵ R}

Notation:

The complex number (a, b) is written as a + ib where I = $\sqrt{-1}$

REMARK:

The set C= {(a,b) /a,b ϵ R }is a field under the operation of addition and multiplication defined by

i) (a ,b) + (c,d) = (a+c, b+d)

ii) (a,b) * (c,d) = (ac-bd ,ad+bc)

Conjugation and absolute value

The transformation states z = x+iy to $\bar{z}$ = x-iy is called complex conjugation.

NOTE:

1) A number is real iff it is equal to its conjugate .

2) Conjugation is involuntary (i.e) $\bar{\bar{z}}$ =z

3).Re(z) = x = $\frac{(x+iy)+(x-iy)}{2} = \frac{z+\bar{z}}{2}$

4).Im(z) = y = $\frac{(x+iy)-(x-iy)}{2i} = \frac{z-\bar{z}}{2i}$

5) i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

ii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

6) $z\bar{z}$ = (x+iy)(x-iy) = $x^2 + y^2$

7) $\sqrt{z\bar{z}}$ is called the modules or the absolute value of z and it is denoted by $|z|$, $|z|^2 = z\bar{z} = x^2+y^2$.

8) i) $|z_1z_2| = |z_1|\,|z_2|$

ii) $|z| = |\bar{z}|$

9) $|z_1+z_2| \leq |z_1| + |z_2|$

$$|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1}+\overline{z_2}) = z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2}$$

$$= |z_1|^2 + |z_2|^2 + 2\mathrm{Re}\,(z_1\overline{z_2})$$

$$\leq |z_1|^2 + |z_2|^2 + 2|z_1\overline{z_2}|$$

$$= |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2|$$

$$|z_1+z_2|^2 \le (|z_1+z_2|)^2$$
$$|z_1+z_2| \le |z_1| + |z_2|$$

10) $|z_1+z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + |z_2|^2$

$$|z_1+z_2|^2 + |z_1 - z_2|^2 = (z_1+z_2)(\overline{z_1}+\overline{z_2}) + (z_1-z_2))(\overline{z_1}-\overline{z_2})$$
$$= |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\overline{z_2}) + |z_1|^2 + |z_2|^2 - 2\mathrm{Re}(z_1\overline{z_2})$$
$$|z_1+z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$$

NOTE:

By induction hypothesis

i) $|z_1+z_2+\ldots\ldots+z_n| \le |z_1| + |z_2| + \ldots\ldots + |z_n|$

ii) $-|z| \le Re(z) \le |z|$

iii) $-|z| \le Im(z) \le |z|$

SQUARE ROOT

Let $\sqrt{\alpha+i\beta} = x+iy$ (α, β *real numbers*)

$\alpha + i\beta = (x+iy)^2 = x^2 - y^2 + 2ixy$

Equating real and imaginary parts

$\propto = x^2 - y^2$ ……………….(1)

$\beta = 2xy$

We have to solve for x and y

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$= \propto^2 + \beta^2$

$x^2+y^2 = \sqrt{\propto^2+\beta^2}$ -----------(2)

(1) + (2) gives $2x^2 = \alpha + \sqrt{\alpha^2+\beta^2}$ and $2y^2 = -\alpha + \sqrt{\alpha^2+\beta^2}$

$$\Rightarrow \quad x = \sqrt{\frac{\alpha+\sqrt{\alpha^2+\beta^2}}{2}} \quad \text{and} \quad y = \sqrt{\frac{-\alpha+\sqrt{\alpha^2+\beta^2}}{2}}$$

The signs of x and y are so chosen that $2xy = \beta$ is satisfied

$\sqrt{\alpha+i\beta} = x + iy$

$$= \pm \sqrt{\frac{\alpha+\sqrt{\alpha^2+\beta^2}}{2}} + i\frac{\beta}{|\beta|}\sqrt{\frac{-\alpha+\sqrt{\alpha^2+\beta^2}}{2}} \text{, provided } \beta \ne 0$$

If β = 0 then square root is $\sqrt{\alpha}$ if $\alpha \geq 0$

Modulus – Amplitude form of a complex number:

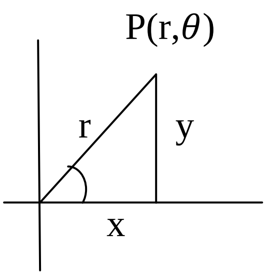

Fig. 3.1

Any complex number can be written of the form z=r (cos Θ + i sin Θ), where Θ = amp z = arg z and r = |z|. Then we have x=r cos Θ and y = r sin Θ and r = |z| and Θ=$\tan^{-1}$(y/x) .

PROBLEM : 1. Show that the area of the triangle with vertices z_1, z_2, z_3 is given by $\sum \frac{|z_1|^2 (z_2 - z_3)}{4iz_1}$.

The area of the Δ ABC = $\frac{1}{2}\sum x_1 (y_2-y_3)$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2i}\begin{vmatrix} x_1 & iy_1 & 1 \\ x_2 & iy_2 & 1 \\ x_3 & iy_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2i}\begin{vmatrix} x_1 + iy_1 & iy_1 & 1 \\ x_2 + iy_2 & iy_2 & 1 \\ x_3 + iy_3 & iy_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2i}\begin{vmatrix} z_1 & \frac{i(z_1 - \overline{z_1})}{2i} & 1 \\ z_2 & \frac{i(z_2 - \overline{z_2})}{2i} & 1 \\ z_3 & \frac{i(z_3 - \overline{z_3})}{2i} & 1 \end{vmatrix}$$

$$= \frac{1}{4i} \begin{vmatrix} z_1 & (z_1 - \overline{z_1}) & 1 \\ z_2 & (z_2 - \overline{z_2}) & 1 \\ z_3 & (z_3 - \overline{z_3}) & 1 \end{vmatrix}$$

$$= \frac{1}{4i} \begin{vmatrix} z_1 & z_1 & 1 \\ z_2 & z_2 & 1 \\ z_3 & z_3 & 1 \end{vmatrix} + \frac{1}{4i} \begin{vmatrix} z_1 & -\overline{z_1} & 1 \\ z_2 & -\overline{z_2} & 1 \\ z_3 & -\overline{z_3} & 1 \end{vmatrix}$$

$$= \frac{1}{4i} \begin{vmatrix} z_1 & -\overline{z_1} & 1 \\ z_2 & -\overline{z_2} & 1 \\ z_3 & -\overline{z_3} & 1 \end{vmatrix} = \frac{1}{4i} \begin{vmatrix} \overline{z_1} & z_1 & 1 \\ \overline{z_2} & z_2 & 1 \\ \overline{z_3} & z_3 & 1 \end{vmatrix}$$

$$= \frac{1}{4i} \sum \overline{z_1}(\, z_2 - z_3)$$

Problem 2: Show that the equation of the circle with centre α (complex) and radius r is $z\bar{z} - \alpha\bar{z} - \bar{\alpha}\, z + |\alpha|^2 - r^2 = 0$.

Solution: Let C be the centre and P(z) be any point on a circle then

$$CP = r \Rightarrow |z-\alpha| = r \Rightarrow |z-\alpha|^2 = r^2 \Rightarrow (z-\alpha)(\overline{z-\alpha}) = r^2$$

$$\Rightarrow \quad z\bar{z} - z\bar{\alpha} - \alpha\bar{z} + \alpha\bar{\alpha} = r^2 \Rightarrow z\bar{z} - \alpha\bar{z} - \bar{\alpha}\, z + |\alpha|^2 - r^2 = 0$$

Problem 3: Prove (i) $\left|\frac{a-b}{1-\bar{a}b}\right| = 1$ if either |a|=1 or |b|=1. when will be the equation true if |a|=|b|=1 ?. (ii) If |a|<1 and |b|<1 then prove that $\left|\frac{a-b}{1-a\bar{b}}\right| < 1$

Solution: (i) Consider $|a| = 1 \Rightarrow a\bar{a} = 1$

Let $w = \frac{a-b}{1-a\bar{b}}$

$$w = \frac{a-b}{a\bar{a}-\bar{a}b} = \frac{a-b}{\bar{a}(a-b)} = \frac{1}{\bar{a}} \Rightarrow \bar{w} = 1/a$$

$$w\bar{w} = (1/a)\,(1/\bar{a}) = \frac{1}{a\bar{a}} = \frac{1}{|a|^2} = 1$$

$$|w|^2 = 1 \Rightarrow |w| = 1$$

$$\left|\frac{a-b}{1-a\bar{b}}\right| = 1$$

Similarly, when |b|=1 we can prove that |w|=1.

$$\therefore \left|\frac{a-b}{1-a\bar{b}}\right| = 1 \text{ if either } |a| = 1 \text{ or } |b| = 1$$

Let |a|= |b|=1

Now $\left|\frac{a-b}{1-a\bar{b}}\right| = \left|\frac{a-b}{a\bar{a}-a\bar{b}}\right| = \frac{|a-b|}{|\bar{a}||a-b|}$

Therefore, $\left|\frac{a-b}{1-a\bar{b}}\right| = 1$ is true only if a≠b

But we have |a| = 1 = |b| and hence the equation is true only when arg a ≠ arg b

(ii) Given |a| < 1 and |b| < 1.

To Prove $|a-b| < |1-\bar{a}b|$ (ie) T.P $|a-b|^2 < |1-\bar{a}b|^2$ (ie) T.P $|a-b|^2 - |1-\bar{a}b|^2 < 0$

Consider $|a-b|^2 - |1-\bar{a}b|^2 = (a-b)(\bar{a}-\bar{b}) - (1-\bar{a}b)(1-a\bar{b})$

$= a\bar{a} - a\bar{b} - b\bar{a} + b\bar{b} - 1 + \bar{a}b + a\bar{b} - a\bar{a}b\bar{b} = |a|^2 - 1 + |b|^2 - |a|^2|b|^2$

$= (|a|^2-1) - |b|^2(|a|^2-1) = (|a|^2-1)(1-|b|^2) < 0$

Hence $\left|\frac{a-b}{1-a\bar{b}}\right| < 1.$

Cauchy's Inequality:

Let a_i, b_i (i = 1, 2,............,n) be complex numbers

$$\left|\sum_{i=1}^{n} a_i\, b_i\right|^2 \leq \left(\sum_{i=1}^{n}(|a_i|)^2\right)\left(\sum_{i=1}^{n}(|b_i|)^2\right)$$

Proof: Let λ be any complex number and we assume that not all b_i's are zero [If all b_i's are zero, then the given in equation is clearly true.

Consider $\sum_{i=1}^{n}\left(|a_i - \lambda\bar{b}_i|^2\right) \geq 0$

(ie) $\sum_{i=1}^{n} |a_i|^2 + |\lambda|^2 \sum_{i=1}^{n} |b_i|^2 - 2\mathrm{Re}\,\bar{\lambda} \sum_{i=1}^{n}(a_i b_i) \geq 0$

This is true for any complex number λ and for any $\lambda = \frac{\sum_{i=1}^{n}(a_i b_i)}{\sum_{i=1}^{n}|b_i|^2}$

Hence we get

$$\sum_{i=1}^{n} |a_i|^2 + \frac{|\sum_{i=1}^{n} a_i b_i|^2}{(\sum_{i=1}^{n}|b_i|^2)^2} \sum_{i=1}^{n}(|b_i|)^2 - 2\mathrm{Re} \frac{|\sum_{i=1}^{n} a_i b_i|^2}{\sum_{i=1}^{n}|b_i|^2} \geq 0$$

$$\sum_{i=1}^{n} |a_i|^2 + \frac{|\sum_{i=1}^{n}(a_i b_i)|^2}{\sum_{i=1}^{n}|b_i|^2} - 2 \frac{|\sum_{i=1}^{n}(a_i b_i)|^2}{\sum_{i=1}^{n}|b_i|^2} \geq 0$$

$$\sum_{i=1}^{n} |a_i|^2 - \frac{|\sum_{i=1}^{n}(a_i b_i)|^2}{\sum_{i=1}^{n}|b_i|^2} \geq 0$$

$$|\sum_{i=1}^{n} a_i b_i|^2 \leq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2$$

Lagrange's Identity

$$|\sum_{i=1}^{n} a_i b_i|^2 = \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \sum_{i<j} |a_i \bar{b_j} - a_j \bar{b_i}|^2$$

where a_1, a_2... a_n and b_1, b_2.... b_n are arbitrary complex numbers. Deduce the Cauchy's Inequality.

Proof:

$\sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2$

$=(a_1\bar{a}_1 + a_2\bar{a}_2 + \dots\dots\dots\dots + a_n\bar{a}_n)(b_1\bar{b}_1 + b_2\bar{b}_2 + \dots\dots\dots\dots + b_n\bar{b}_n)$

$= a_1\bar{a}_1 b_1\bar{b}_1 + a_2\bar{a}_2\, b_2\bar{b}_2 + \dots\dots\dots\dots + a_n\bar{a}_n b_n\bar{b}_n + a_1\bar{a}_1\, b_2\bar{b}_2 + a_1\bar{a}_1\, b_3\bar{b}_3 + \dots\dots\dots + a_1\bar{a}_1\, b_n\bar{b}_n + a_2\bar{a}_2\, b_3\bar{b}_3 + a_2\bar{a}_2\, b_4\bar{b}_4 + \dots\dots\dots\dots + a_2\bar{a}_2\, b_n\bar{b}_n + \dots\dots\dots\dots + a_{n-1}\bar{a}_{n-1}\, b_n\bar{b}_n + b_1\bar{b}_1\, a_2\,\bar{a}_2 + \dots\dots\dots + b_1\bar{b}_1\, a_n\bar{a}_n + b_2\bar{b}_2\, a_3\bar{a}_3 + \dots\dots\dots + b_2\bar{b}_2\, a_n\bar{a}_n + \dots\dots\dots\dots + b_{n-1}\bar{b}_{n-1}\, a_1\bar{a}_1$

$= |\sum_{i=1}^{n} a_i b_i|^2 + \sum_{1\leq i<j\leq n} |a_i \bar{b_j}|^2 + \sum_{1\leq i<j\leq n} |a_j \bar{b_i}|^2$ ……..(1)

We know that $|a-b|^2 = |a|^2 + |b|^2 - 2\,\mathrm{Re}\,(a\bar{b})$

$\sum_{1\le i<j\le n} |a_i\bar{b_j} - a_j\bar{b_i}|^2 = \sum_{i<j} |a_i\bar{b_j}|^2 + \sum_{i<j} |a_j\bar{b_i}|^2 - 2\mathrm{Re}\ \sum a_i\,\bar{b_j}\,\bar{a_j}b_i$ ----(2)

(1) - (2)

$\sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \sum_{i<j} |a_i\bar{b_j} - a_j\bar{b_i}|^2$

$= \sum_{i=1}^{n} |a_i b_i|^2 + \sum_{i<j} |a_i\bar{b_j}|^2 + \sum_{i<j} |a_j\bar{b_i}|^2 - \sum_{i<j} |a_i\bar{b_j}|^2 - \sum_{i<j} |a_j\bar{b_i}|^2$

$+2\mathrm{Re} \sum_{i<j} a_i\bar{b_j}\,\bar{a_j}b_i$

$= |\sum_{i=1}^{n} a_i b_i|^2 + \mathrm{Re} \sum_{i<j} a_i\bar{b_j}\,\bar{a_j}b_i$ ……………..(3)

Now, $|\sum_{i=1}^{n} a_i b_i|^2 = (a_1b_1 + a_2b_2 + \dots + a_nb_n) + (\bar{a}_1\bar{b}_1 + \bar{a}_2\bar{b}_2 + \dots + \bar{a}_n\bar{b}_n)$

$= a_1b_1\,\bar{a}_1\bar{b}_1 + a_2b_2\,\bar{a}_2\bar{b}_2 + \dots + a_nb_n\,\bar{a}_n\bar{b}_n + a_1b_1\,\bar{a}_2\bar{b}_2 + a_1b_1$

$\bar{a}_3\bar{b}_3 + \dots + a_1b_1\,\bar{a}_n\bar{b}_n + \dots + a_{n-1}b_{n-1}\,\bar{a}_n\bar{b}_n + \bar{a}_1\bar{b}_1$

$a_2b_2 + \dots + \bar{a}_1\bar{b}_1\,a_nb_n + \bar{a}_2\bar{b}_2\,a_3b_3 + \dots + \bar{a}_2\bar{b}_2\,a_nb_n + \dots$

$+ \bar{a}_{n-1}\bar{b}_{n-1}\,a_nb_n$

$= |\sum_{i=1}^{n} a_i b_i|^2 + \sum_{i<j} a_ib_i\bar{a_j}\bar{b_j} + \sum_{i<j} a_jb_j\bar{a_i}\bar{b_i}$

$= |\sum_{i=1}^{n} a_i b_i|^2 + 2\mathrm{Re} \sum_{i<j} a_ib_i\bar{a_j}\bar{b_j}$……….(4)

From (3) and (4)

$|\sum_{i=1}^{n} a_i b_i|^2 = \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \sum_{1\le i<j\le n} |a_i\bar{b_j} - a_j\bar{b_i}|^2$

Deduction:

Since $\sum_{1\le i<j\le n} |a_i\bar{b_j} - a_j\bar{b_i}|^2 \ge 0$

$-\sum_{1\le i<j\le n} |a_i\overline{b_j} - a_j\overline{b_i}|^2 \le 0$

(ie) $\left|\sum_{i=1}^{n} a_i b_i\right|^2 \le \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2$, which is the required Cauchy inequality.

NOTE:

The equation of a straight line can be written as z = a+bt, where t is real.

$\Rightarrow \frac{z-a}{b}$ = t = real $\Rightarrow$Im $\frac{z-a}{b}$ = 0. If Im $\frac{z-a}{b} < 0$ then it is right half plane and if Im $\frac{z-a}{b} > 0$ then it is left half plane.

Spherical Representation

The system C of complex numbers can be extended by introducing the symbol ∞. Now its connection with finite numbers is given by $a+\infty = \infty+a = \infty\ \forall$ finite a and $b.\infty = \infty.b = \infty\ \forall\ b \ne 0$ including $b = \infty$.

Further, $\infty + \infty$ and $0.\infty$ are not defined

$\frac{a}{0} = \infty\ \forall\ a \ne 0$, $\frac{a}{\infty} = 0\ \forall\ b \ne \infty$. We call ∞ as the point at ∞.

Extended complex plane

The points in the plane together with the point at ∞ form the extended complex plane.

1. Every straight line shall pass through the at ∞ (Ideal point)

2. No half plane shall contain the ideal point.

Stereographic projection

It is a geometric model in which all points of the extended plane we have a concrete representation.

Consider the unit sphere S whose equation is $x_1^2 + x_2^2 + x_3^2 = 1$ with every point on S except (0, 0, 1) we can associate a complex number $z = \frac{x_1 + ix_2}{1 - x_3}$ and this correspondence is 1-1.

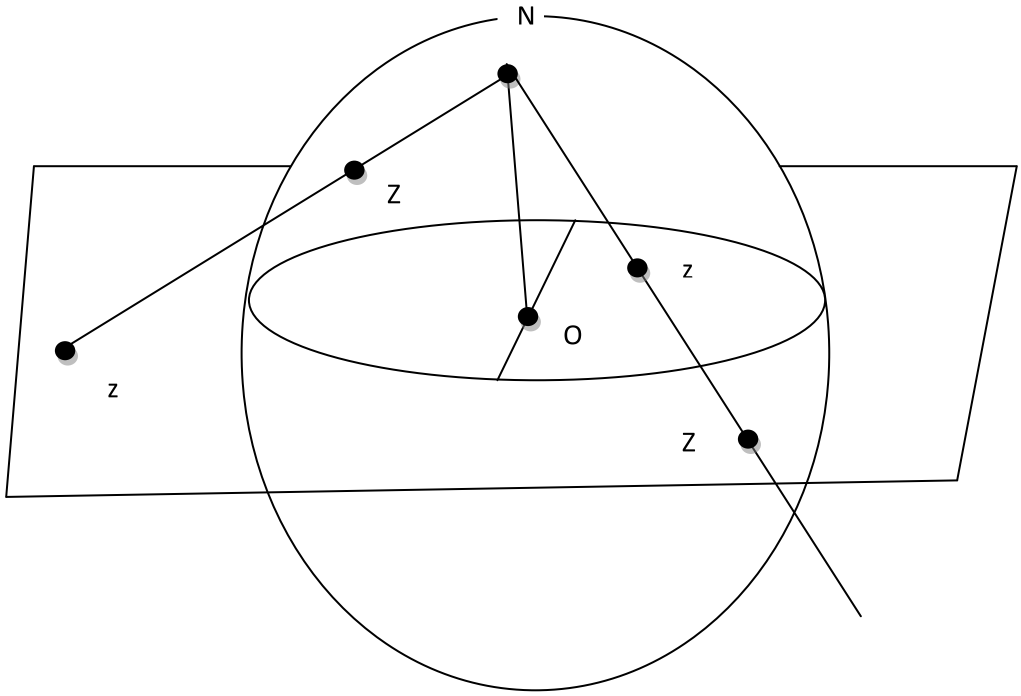

FIG. 3.2 Stereographic projection.

Now, $|z|^2 = \frac{{x_1}^2 + {x_2}^2}{(1-x_3)^2} = \frac{1-{x_3}^2}{(1-x_3)^2}$ [as $x_1^2 + x_2^2 + x_3^2 = 1$]

$$= \frac{1+x_3}{1-x_3} \Rightarrow x_3 = \frac{|z|^2-1}{|z|^2+1}$$

Similarly, $x_1 = \frac{z + \bar{z}}{|z|^2+1}$ and $x_2 = \frac{z - \bar{z}}{i(|z|^2+1)}$

The correspondence can be completed by letting the point at ∞ corresponds to (0, 0, 1).

$\therefore$ We can regard the sphere as a representation of the extended plane or of the extended number system.

Note that the hemisphere $x_3 < 0$ corresponds to the disc $|z| < 1$ and the hemisphere $x_3 > 0$ corresponds to its outside $|z| < 1$.

In function theory the sphere 'S' is referred as the Riemann sphere.

If the complex plane is identified with (x_1, x_2) plane with x_1 axis and x_2 axis corresponding to real and Imaginary axis respectively then the transformation $z = \frac{x_1+ix_2}{1-x_3}$ takes on a simple geometrical meaning. Now writing $z = x + iy$

We have $x + iy = \frac{x_1+ix_2}{1-x_3}$

Equating real and Imaginary part

$$x = \frac{x_1}{1-x_3} \quad , \quad y = \frac{x_2}{1-x_3}$$

$$\frac{x}{x_1} = \frac{y}{x_2} = \frac{-1}{x_3 - 1} \qquad \text{(or)} \quad x : y : -1 = x_1 : x_2 : x_3 - 1$$

The points $(x, y, 0)$, (x_1, x_2, x_3) and $(0, 0, 1)$ are in a straight line.

Hence the correspondence is a central projection from the center $(0, 0, 1)$. It is called a stereographic projection. It is geometrically evident that stereographic projection transforms every straight line in the z-plane into the circle on S which passes through the pole $(0, 0,1)$ and the converse is also.

More generally, any circle on the sphere corresponds to circle or the straight line z-plane. To prove this we observe that a circle on the sphere lies in a plane $\alpha_1x_1+ \alpha_2x_2+ \alpha_3x_3 = \alpha_0$ and $\alpha_1^2+ \alpha_2^2+ \alpha_3^2 = 1$ and $0 \le \alpha_0 < 1$

$$\alpha_1 \left(\frac{z+\bar{z}}{|z|^2+1}\right) + \alpha_2 \left(\frac{z-\bar{z}}{i(|z|^2+1)}\right) + \alpha_3 \left(\frac{|z|^2-1}{|z|^2+1}\right) = \alpha_0$$

$$\alpha_1(z + \overline{z}) - i\alpha_2(z - \overline{z}) + \alpha_3(|z|^2 - 1) = \alpha_0(1 + |z|^2)$$

$$\alpha_1 2x - i\alpha_2(2iy) + (\alpha_3 - \alpha_0) |z|^2 = \alpha_0 + \alpha_3$$

$$(x^2 + y^2)(\alpha_3 - \alpha_0) + 2 \alpha_1 x + 2 \alpha_2 y - (\alpha_0 + \alpha_3) = 0$$

For $\alpha_0 \neq \alpha_3$, this is the equation of the circle.

For $\alpha_0 = \alpha_3$, it represents as a straight line.

Conversely, the equation of any circle or straight line can be written in this form.

This correspondence is consequently one to one.

To calculate the distance $d(z,z')$ between the stereographic projection of z and z′.

The points on the sphere are denoted by (x_1, x_2, x_3) and (x_1', x_2', x_3')

$$d(z,z') = \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2}$$

$$= \sqrt{2 - 2(x_1x_1' + x_2x_2' + x_3x_3')}$$

Consider,

$$x_1\, x_1' + x_2\, x' + x_3\, x_3' = \frac{(z+\bar{z})(z'+\bar{z}') - (z-\bar{z})(z'-\bar{z}') + (|z|^2-1)(|z'|^2-1)}{(1+|z|^2)\,(1+|z'|^2)}$$

$$= \frac{zz' + z\bar{z}' + \bar{z}z + \bar{z}\bar{z}' - zz' + z\bar{z}' + \bar{z}z' - \bar{z}\bar{z}' + (|z|^2-1)\,(|z'|^2-1)}{(1+|z|^2)\,(1+|z'|^2)}$$

$$= \frac{2\,(z\bar{z}' + \bar{z}z') + |z|^2|z'|^2 - |z|^2 - |z'|^2 + 1}{(1+|z|^2)\,(1+|z'|^2)}$$

$$= \frac{(|z|^2+1)\left(1 + |z'|^2\right) - |z|^2|z'|^2 - |z|^2 - |z'|^2 - 1 + 2\,(z\bar{z}' + \bar{z}z') + |z|^2|z'|^2 - |z|^2 - |z'|^2 + 1}{(1+|z|^2)\,(1+|z'|^2)}$$

$$= 1 - \frac{2\,(|z|^2 + |z'|^2 - (z\bar{z}' + \bar{z}z'))}{(1+|z|^2)(1+|z'|^2)}$$

$$= 1 - \frac{2|z-z'|^2}{(1+|z|^2)\,(1+|z'|^2)}$$

$\therefore d(z,z') = \sqrt{2-2\left(1-\frac{2\,|z-z'|^2}{(1+|z|^2)(1+|z'|^2)}\right)} \quad = \sqrt{\frac{4\,|z-z'|^2}{(1+|z|^2)\,(1+|z'|^2)}} = \frac{2\,|z-z'|}{\sqrt{(1+|z|^2)(1+|z'|^2)}}$

For $z' = \infty$ the corresponding formula is

$d(z,\infty) = \sqrt{(x_1-0)^2+(x_2-0)^2+(x_3-1)^2}$

$$=\sqrt{(x_1^2+x_2^2+x_3^2)+1-2x_3} = \sqrt{2-2x_3} \quad = \sqrt{2(1-x_3)}$$

$$= \sqrt{2}\sqrt{1-\frac{(|z|^2-1)}{(1+|z|^2)}} \quad = \sqrt{2}\sqrt{1-\frac{(1+|z|^2-2)}{1+|z|^2}}$$

$$= \sqrt{2}\sqrt{1-\left(1-\frac{2}{1+|z|^2}\right)} = \sqrt{2}\sqrt{\frac{2}{1+|z|^2}} \;=\; \frac{2}{\sqrt{1+|z|^2}}$$

Problem 1 Show that z and z' corresponds to diametrically opposite points to Riemann's sphere iff $z\,\overline{z'} = -1$.

Solution: Let the diametrically opposite points be (α, β, γ) and $(-\alpha, -\beta, -\gamma)$

$$z = \frac{\alpha+i\beta}{1-\gamma} \quad ; \quad z' = \frac{-\alpha-i\beta}{1+\gamma}$$

$z\bar{z}' = \frac{\alpha+i\beta}{1-\gamma}\frac{(-\alpha+i\beta)}{(1+\gamma)} \qquad = \frac{-\beta^2-\alpha^2}{1-\gamma^2} \qquad = \frac{\gamma^2-1}{1-\gamma^2} = -1$ [as $\alpha^2+\beta^2+\gamma^2 = 1$]

Conversely, $\qquad z\bar{z}' = -1$

Let $z = \frac{\alpha+i\beta}{1-\gamma}$. Since $z\bar{z}' = -1$, $\quad (\frac{\alpha+i\beta}{1-\gamma})\bar{z}' = -1$

$$\bar{z}' = \frac{-(1-\gamma)}{\alpha+i\beta)} \;, z' = \frac{-(1-\gamma)}{\alpha-i\beta} = \frac{-(1-\gamma)}{(\alpha-i\beta)}\frac{(\alpha+i\beta)}{(\alpha+i\beta)} = \frac{-(1-\gamma)(\alpha+i\beta)}{\alpha^2+\beta^2}$$

$$= \frac{-(1-\gamma)(\alpha+i\beta)}{1-\gamma^2} = \frac{-(1-\gamma)(\alpha+i\beta)}{(1-\gamma)(1+\gamma)}$$

$$z' = \frac{-\alpha-i\beta}{1+\gamma}$$

Therefore , z = (α, β, γ) then $z' = (-\alpha, -\beta, -\gamma)$

Therefore , z and z′ are diametrically opposite points.

3.2 Analytic Functions

Introduction to the concept of analytic function

There are four different types of functions

1. Real function of a complex variable
2. Complex function of a real variable
3. Real functions of a real variable
4. Complex functions of a complex variable

Notation:

W = f(z) is to denote complex function of a complex variable for the remaining three functions, we use y = f(x), where x and y be real or complex. If a variable is definitely restricted by real values, then we denote it by t. All functions must be defined and consequently single valued.

Limit and Continuity:

Definition :

The function f(x) is said to have the limit A as x tends to a.

$\lim_{x\to a} f(x) = A \to (1)$ if and only if the following is true

For every $\varepsilon > 0$,there exists a number $\delta > 0$ with the property that

$|f(x)-A| < \varepsilon$ for all values of x such that $|x-a| < \delta$ and $x \neq a$.

Form eqn (1), $\lim_{x\to a} \overline{f(x)} = \bar{A} \to (2)$

From (1) and (2), $\lim_{x\to a} \mathrm{Re}(f(x)) = \mathrm{Re}(A) \to (3a)$

Similarly, $\lim_{x\to a}$ Im(f(x)) =Im(A)→ $(3b)$

Conversely, (1) is a consequence of equation (3a) and (3b).

Definition:

The function f(x) said to be continuous at x=a iff $\lim_{x\to a}$ f(x)=f(a).

f is continuous iff f is continuous at all points where it is defined. The sum and product of two continuous functions are continuous. The quotient $\frac{f(x)}{g(x)}$ is defined and continuous at a, provided g(a) ≠0.

If f is continuous, so are Re(f(x)), Im(f(x)) and |f(x)| is continuous.

The derivative of a function:

$$f''(a) = \lim_{x\to a} \frac{f(x)- f(a)}{x-a}$$

The usual result for forming the derivative of a sum , a product or a quotient are all valid. The derivative of a composite function is determined by the chain rule.

There is a fundamental difference between the cases of a Real and Complex Independent variable.

Result: The real function of a complex variable either has a derivative zero or else the derivative does not exist.

Proof: Let f(x) be real function of a complex variable whose derivative exists at z = a . Then f′(a) is on one side is real, for it is the limit of the quotient $\frac{f(a+h)-f(a)}{h}$ as h→ 0 through all real values, On the other side, it is also the limit of the quotient $\frac{f(a+ih)-f(a)}{ih}$ and as such purely Imaginary.

Since f′(a) exists and is unique and it is both real and Imaginary ⇒ f′(a) = 0

Example: W = f(z) = $|z|^2$

$\Delta w = |z+\Delta z|^2 - |z|^2 \ = (z+\Delta z)\,(\bar{z}+\overline{\Delta z}) - z\bar{z} \ = \ z\bar{z}+z\overline{\Delta z}+\bar{z}\Delta z+\Delta z\overline{\Delta z} - z\bar{z}$

$\frac{\Delta w}{\Delta z} = \frac{z\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z}$ .... (1)

When $z = 0$, $\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z} = 0 \Rightarrow \frac{dw}{dz} = 0$

When $z \neq 0$, Let $\Delta z \to 0$ through all real values

Take $\Delta z = h$, $\overline{\Delta z} = h$. Then , $\frac{\Delta w}{\Delta z} = z + \bar{z} + h$

$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = z + \bar{z} \to (2)$

$\Delta z \to 0$ through purely imaginary values, $\Delta z = ih$ and $\overline{\Delta z} = -ih$

When $\Delta z \to 0 \Rightarrow h \to 0$

From (1) , $\frac{dw}{dz} = -z + \bar{z}$ ....(3)

From (2) and (3), $\frac{dw}{dz}$ does not exist when $z \neq 0$, since the limit is unique.

Therefore, $\frac{dw}{dz}$ exists only at the origin.

The case of a complex function of a real variable can be reduced to the real case.

$z(t) = x(t) + iy(t) \ \Rightarrow z'(t) = x'(t) + y'(t)$

The existence of $z'(t)$ is equivalent to the simultaneously existence of $x'(t)$ and $y'(t)$.

Analytic Function:

The class of analytic function is formed by the complex functions of a complex variable which posses a derivative whenever the function is defined.

The sum and product of two analytic functions is again analytic. The same is true for the quotient $\frac{f(z)}{g(z)}$ of two analytic functions, provided that g(z) also not vanish. In general case, it is necessary to exclude the points at which g(z) = 0.

The definition of the derivative can be written in the form

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} .$$

As a first consequence, f(z) is necessarily continuous .

For, $\quad f(z+h) - f(z) = \frac{h[f(z+h) - f(z)]}{h}$

$$\lim_{h\to 0}(f(z+h) - f(z) \quad = \lim_{h\to 0} \frac{h(f(z+h)-f(z))}{h} = 0f'(z) = 0.$$

Therefore, $\lim_{h\to 0} f(z+h) = f(z)$. Therefore, in general the converse is not true.

Example: f(z) = $|z|^2$. It is continuous at all the points. But it is not differentiable when $z \neq 0$.

If f(z) = u(z) + iv(z) is continuous then it implies u(z) and v(z) are both continuous.

Theorem 1: Let w = f(z) = u(x,y)+iu(x,y) be differentiable at any point in a region D . Then the partial derivatives u_x,u_y and v_x,v_y exist and satisfy the Cauchy Riemann equations $u_x = v_y$, $u_y = -v_x$ (i.e) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Proof:

Let f(z) = u(x,y) + iv(x,y) be analytic at any point z of the region D.

Therefore, $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists and is unique . (i.e) It is independent of the path along which $h \to 0$.

If h = Δx then $f'(z) = \lim_{\Delta x\to 0} \frac{u(x+\Delta x,y)-u(x,y)}{\Delta x} + i \lim_{\Delta x\to 0} \frac{v(x+\Delta x,y)-v(x,y)}{\Delta x}$

$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad (= \frac{\partial f}{\partial x})$

Therefore, $f'(z) = u_x + iv_x \rightarrow (1)$

Since $f'(z)$ exists, the above limit exists which means that u_x and v_x exist.

If $h = i\Delta y$, $f'(z) = \lim_{\Delta y \to 0} \frac{u(x,y+i\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x,y+i\Delta y) - v(x,y)}{i\Delta y}$

$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial f}{\partial y}$

$f'(z) = -i\, u_y + v_y \rightarrow (2)$

Since $f'(z)$ exists, the above limit exists which means that u_y and v_y exist

Since the limit should be unique, from (1) and (2) $u_x + iv_x = -i\, u_y + v_y$

Equating real and imaginary parts, we have $u_x = v_y$ and $u_y = -v_x$

These are called C-R equations.

The following theorem is the sufficient condition for function is to be analytic.

Theorem 2: If u(x,y) and v(x,y) have continuous first order partial derivative which satisfy the C-R equations , then f(z) = u(z) + i v(z) is analytic with continuous derivative $f'(z)$.

Proof: Let f(z) = u(x,y) + iv(x,y) where $u_x = v_y$, $u_y = -v_x$

Now, f(z+h+ik) - f(z) = f(x+iy+h+ik) - f(x+iy) = f(x+h+i(y+k)) - f(x+iy)

= f(x+h+i(y+k)) - f(x+iy)

= u(x+h,y+k) + i v(x+h,y+k) - u(x,y) - i v(x,y)

= u(x+h,y+k) - u(x+h,y) + u(x+h,y) - u(x,y)

+i [v(x+h,y+k) - v(x+h,y) + v(x+h,y) - v(x,y)]

Using mean value theorem we get,

u(x+h , y+k) - u(x+h , y) = k u_y(x + h , y + θ_1k) $\quad$ (0 < θ_1< 1) $\quad$ [$\because$ u_y exist]

= k(u_y(x,y) + λ_1) where $\lambda_1 \to 0$ as h$\to$ 0, k$\to$ 0 [$\because$ u_y is continuous]

u(x+h ,y) – u(x , y) = hu_x(x+hθ_2 , y) $\quad$ (0<θ_2<1) $\quad$ [$\because$ u_x exist]

= h[u_x((x,y)+λ_2] where $\lambda_2 \to$ 0 as h$\to$0, k$\to$0 [$\because$ u_x is continuous]

u(x+h , y+k) – u(x , y) = h $\frac{\partial u}{\partial x}$ +k $\frac{\partial u}{\partial y}$ + hλ_2+ kλ_1

= h $\frac{\partial u}{\partial x}$ + k $\frac{\partial u}{\partial y}$ + $\mathcal{E}_1$ $\quad$ where $\quad$ $\mathcal{E}_1$ = λ_1k + hλ_2 &

where $\lambda_{1,}\lambda_2 \to$ 0 as h$\to$0, k$\to$0

Similarly,

v(x+h , y+k) – v(x , y) = h $\frac{\partial v}{\partial x}$ + k $\frac{\partial v}{\partial y}$ + $\mathcal{E}_2$ $\quad$ where $\quad$ $\mathcal{E}_2$ = kλ_3 + hλ_4

where $\lambda_3 \to 0$, $\lambda_4 \to 0$ as h$\to$0 , k$\to$0

Taking limit h+ik $\to$0

$\frac{k\lambda_1}{h+ik}$, $\frac{h\lambda_2}{h+ik}$, $\frac{k\lambda_3}{h+ik}$ and $\frac{h\lambda_4}{h+ik} \to 0$ $\quad$ [as h$\to$ 0,k$\to$ 0, and $\lambda_{1,}\lambda_{2,}\lambda_{3,}\lambda_4 \to 0$]

$\therefore$ $\frac{\varepsilon_1}{h+ik} \to 0$, $\frac{\varepsilon_2}{h+ik} \to 0$ as h+ik$\to$ 0

$\therefore$f(z+h+ik) – f(z) = u(x+h , y+k) – u(x , y) + i(v(x+h , y+k) – v(x , y))

= h $\frac{\partial u}{\partial x}$ + k $\frac{\partial u}{\partial y}$ + $\mathcal{E}_1$ + i (h $\frac{\partial v}{\partial x}$ + k $\frac{\partial v}{\partial y}$ + $\mathcal{E}_2$)

= h ($\frac{\partial u}{\partial x}$ + i $\frac{\partial v}{\partial x}$) + k ($\frac{\partial u}{\partial y}$ + i $\frac{\partial v}{\partial y}$) + $\mathcal{E}_1$ + i $\mathcal{E}_2$

$= h\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + k\left(\frac{-\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right) + \varepsilon_1 + i\,\varepsilon_2$ [$\because u_x = v_y, u_y = -v_x$]

$= h\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + ik\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \varepsilon_1 + i\,\varepsilon_2$

$= (h + ik)\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \varepsilon_1 + i\,\varepsilon_2$

Hence, $\lim_{h+ik\to 0}\frac{f(z+h+ik)-f(z)}{h+ik} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \lim_{h+ik\to 0}\frac{\varepsilon_{1+}\varepsilon_2}{h+ik}$

$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

Since u_x and v_x exist and are unique, $f\,'$(z) exists.

Hence f(z) is analytic at an arbitrary point z. ∴ It is analytic in a region.

Hence the theorem is proved.

It is observed that as the C.R equations are necessary condition for differentiability, if they are not satisfied at a point then the function is not differentiable at that point.

Note that as f '(z) = $u_x + iv_x$ then $|f'(z)|^2 = |u_x + iv_x|^2$

$= u_x^2 + v_x^2$

$= v_y u_x + v_x(-u_y)$ [$\because u_y = -v_x$, $u_x = v_y$]

$= u_x v_y - u_y v_x$

We shall prove later, the derivative of an analytic function is itself analytic.

⇒ u and v will have continuous partial derivatives of all order and in particular the mixed derivatives will be equal.

From the C-R equation, $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow u_{xx} = v_{yx}$ and $u_{yy} = -v_{xy}$

$\Rightarrow u_{xx} = -u_{yy}$ [$\because v_{xy} = v_{yx}$]

$\Rightarrow u_{xx} + u_{yy} = 0$

(i.e) $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Similarly, $\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

A function u satisfies the Laplace's equation $\Delta u = 0$ is said to be harmonic.

∴The real and imaginary parts of an analytic functions are harmonic.

If two harmonic function u and v satisfies the C-R equations then v is said to be harmonic conjugate to u.

If v is a harmonic conjugate of u then $-u$ is the harmonic conjugate of v and conversely.

It is also true that the harmonic conjugate is unique except for an additive constant.

Observation: f(z) is analytic function on D if and only if v is harmonic conjugate of u.

If f(z) = u(x,y) + iv(x,y) is analytic.

⇒ u and v satisfy the C-R equations.

⇒ v is a harmonic conjugate of u.

Conversely

If v is a harmonic conjugate of u , by theorem 2 , the function f(z) = u(x,y) + iv(x,y) is analytic.[for this purpose, we make the explicitly that u and v have continuous first order partial derivatives]

Example 1:

Find the harmonic conjugate of a harmonic function u(x,y)=$x^2 - y^2$.

$u_x = 2x$, $u_y = -2y$

Using C-R equations, $v_y = 2x$ and $-v_x = -2y$ ⇒ $v_x = 2y$

Consider $v_x = 2y$. Integrating w.r.to x keeping y as constant

$v = 2yx + \phi(y)$ ⇒ $v_y = 2x + \varphi'(y)$

⇒ $2x = 2x + \varphi'(y)$ ⇒ $\varphi'(y) = 0$

⇒ $\varphi(y) = c$ (a constant)

$\therefore v = 2xy + c$

f(z) = u+iv $= x^2 - y^2 + i(2xy + c)$

$= x^2 - y^2 + i2xy + ic$

$=(x + iy^2) + ic \;=\; z^2 + ic$

Example 2:

Consider a complex function f(x,y) of two real variables. Let z = x + iy , $\bar{z}$ = x - iy and x = $\frac{z+\bar{z}}{2}$, y = -i($\frac{z-\bar{z}}{2}$). With the change of variables we can consider f(x,y) as a function of z and $\bar{z}$ which will be treated as independent variables.

Soln: $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial z}$

$= \frac{\partial f}{\partial x}\frac{1}{2} + \frac{\partial f}{\partial y}\left(\frac{-i}{2}\right) \quad = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\,\frac{\partial f}{\partial y}\right)$

$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}$

$= \frac{\partial f}{\partial x}\frac{1}{2} + \frac{\partial f}{\partial y}\left(\frac{i}{2}\right) \quad = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\,\frac{\partial f}{\partial y}\right)$

If f is analytic then $\frac{\partial f}{\partial x} = -i\,\frac{\partial f}{\partial y}$

$\Rightarrow \quad \frac{\partial f}{\partial x} + i\,\frac{\partial f}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial \bar{z}} = 0$

$\Rightarrow$ any analytic function is independent of $\bar{z}$ and a function z alone.

Corollary 3:

This formal reasoning supports that analytic functions are true functions of a complex variable as opposed to functions which are more adequately described as complex functions of two real variables.

By similar formal arguments, we derive a simple method which allows to compute without the use of integration.

The analytic function f(z) whose real part is given harmonic function u(x,y) [given a harmonic function u without the use of integration we are now going to determine the analytic function f(z)].

Note that $\frac{\partial \bar{f}}{\partial z} = 0 \Rightarrow \overline{f(z)}$ may be considered as a function of $\bar{z}$, denote it by $\bar{f}(\bar{z})$

$$\Rightarrow u(x,y) = \frac{1}{2} [\ f(z) + i\ \overline{f(z)}]$$

$$= \frac{1}{2} [\ f(z) + i\ \bar{f}(\bar{z})]$$

$$= \frac{1}{2} [\ f(x+iy) + i\ \bar{f}(x-iy)]$$

This is a formal identity. ∴It is reasonable to expect that it holds even when x and y are complex.

Let $x = z/2$, $y = z/2i$

$u(z/2 \ , z/2i) = \frac{1}{2}[f(z) + \bar{f}(0)]$ ……………… (1)

Since f(z) is only determined upto a purely imaginary constant we may as well assume that f(0) is real.

$\Rightarrow \bar{f}(0) = u(0,0)$

$\therefore f(z) = 2u\ (z/2 \ , z/2i) - u(0,0)$ [By (1)]

A pure imaginary constant can be added at will.

NOTE: In this form,the method is definitely related to the function u(x,y) which are rational in x and y for the function must have the meaning for the complex values of the arguments.

Example 3: Show that the harmonic function satisfies the formal differential equation $\frac{\partial^2 u}{\partial z \partial \bar{z}}$.

Soln: Given, u is a harmonic function. $\Rightarrow \Delta u = 0$ $\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Now, $\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z}[\frac{\partial u}{\partial \bar{z}}] = \frac{\partial}{\partial z}[\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}]$

$$= \frac{\partial}{\partial z}[\frac{\partial u}{\partial x} \cdot \frac{1}{2} + \frac{\partial u}{\partial y}(\frac{-1}{2i})] = \frac{1}{2}\frac{\partial}{\partial z}[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}]$$

$$= \frac{1}{2}[\frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial x}{\partial z} + \frac{\partial^2 u}{\partial x \partial y} \cdot \frac{\partial y}{\partial z} + i(\frac{\partial^2 u}{\partial x \partial y} \cdot \frac{\partial x}{\partial z} + \frac{\partial^2 u}{\partial y^2} \cdot \frac{\partial y}{\partial z}]$$

$$= \frac{1}{2}[\frac{\partial^2 u}{\partial x^2} \cdot \frac{1}{2} + \frac{\partial^2 u}{\partial x \partial y} \cdot \frac{1}{2i} + i\frac{\partial^2 u}{\partial x \partial y} \cdot \frac{1}{2} + i\frac{\partial^2 u}{\partial y^2} \cdot \frac{1}{2i}]$$

$$= \frac{1}{4}[\frac{\partial^2 u}{\partial x^2} - i\frac{\partial^2 u}{\partial x \partial y} + i\frac{\partial^2 u}{\partial x \partial y} . + \frac{\partial^2 u}{\partial y^2}]$$

$$= \frac{1}{4}[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}] = \frac{1}{4} . 0 = 0$$

Aliter: $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

Theorem 4:

If f(z) = u +iv be an analytic function in a region D,then prove that f(z) is constant in D. If any one of the following conditions hold ,

(i) f ’(z) vanishes identically in D. (ii) R[f(z)] = u = constant.

(iii) Im(f(z)) = v = constant. (iv) $|f(z)|$ = constant.

(v)arg f(z) = constant.

Proof: Now f(z) =u +iv

f ‘(z) = $\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$ = $\frac{\partial v}{\partial y}$ - $i\,\frac{\partial u}{\partial y}$ (By C-R eqn)

(i) Now, f ‘(z) ≡ 0 ⇒ $u_x + iv_x \equiv 0$ ⇒ $v_y - iu_y \equiv 0$

⇒ u_x = 0 , v_x = 0 , u_y = 0 , v_y = 0

⇒ u and v are constant on any line segments parallel to co-ordinate axis. But any two points in D can be joined by such parallel lines.

⇒ f(z) is constant.

(ii) u = a is constant ⇒ $\frac{\partial u}{\partial x}$ = 0 , $\frac{\partial u}{\partial y}$ = 0

f ‘(z) = $\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$ = $\frac{\partial u}{\partial x}$ - $i\,\frac{\partial u}{\partial y}$ = 0

By (i), f(z) is constant.

(iii) v = constant ⇒ $\frac{\partial v}{\partial x}$ = 0 , $\frac{\partial v}{\partial y}$ = 0

f ‘(z) = $\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\,\frac{\partial v}{\partial x}$ = 0

By (i), f(z) is constant.

(iv) $|f(z)|$ = constant ⇒ $u^2 + v^2$ = constant

⇒ $2u\frac{\partial u}{\partial x} + 2v\,\frac{\partial v}{\partial x}$ = 0 …………..(1)

$2u\frac{\partial u}{\partial y} + 2v\,\frac{\partial v}{\partial y}$ = 0 …………..(2)

u X (1) + v X (2)

⇒ $u^2\frac{\partial u}{\partial x} + u\,v\,\frac{\partial v}{\partial x} + uv\frac{\partial u}{\partial y} + v^2\,\frac{\partial v}{\partial y}$ = 0

$\Rightarrow (u^2 + v^2)\dfrac{\partial u}{\partial x} = 0 \quad [as\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x}]$

$\Rightarrow u^2 + v^2 = 0$ (or) $\dfrac{\partial u}{\partial x} = 0$

Similarly,

$\Rightarrow u^2 + v^2 = 0$ (or) $\dfrac{\partial v}{\partial x} = 0$

If $u^2 + v^2 \neq 0$ then u and v are constant

Therefore, $f(z)$ is constant.

If $u^2 + v^2 = 0$ at a point and it is constantly zero and $f(z) = 0$.

(v) arg $f(z)$ = c = constant

$\Rightarrow \tan^{-1}\left(\frac{v}{u}\right) = c \quad \Rightarrow \frac{v}{u} = \tan c \quad \Rightarrow v = u\tan c \quad \Rightarrow u = \cot c\ v$

$= kv$ where $k = \cot c \Rightarrow u - kv = 0 \Rightarrow Re((1 + ik)(u + iv)) = 0$

$\Rightarrow Re((1 + ik)f(z)) = 0 \quad \Rightarrow (1 + ik)f(z)$ = constant [By (ii)]

$\Rightarrow$ f(z) = constant

Polynomials:

Every constant is a analytic function with derivative zero. The simple non constant analytic function is z whose derivative is one. Since the sum and the product of two analytic functions are again analytic $\Rightarrow$ every polynomial p(z) = $a_0 + a_1 z + \ldots\ldots\ldots + a_{n_{z^n}}$ is an analytic function and its derivative f'(z) = $a_1 + \ldots\ldots\ldots\ldots + na_{n_{z^{n-1}}}$ is analytic. If $a_n \neq 0$ then deg p(z) = n.

For formal reason the constant zero is regarded as a polynomial and its degree is $-\infty$. Therefore, the zero polynomial is excluded from our consideration.

By fundamental theorem of Algebra, P(z) = 0 has at least one root for n $\geq$ 0.

If P(α_1) = 0 $\Rightarrow$ P(z) =(z- α_1)P_1(z) where P_1(z) is the polynomial of degree n-1. The repetition of this process leads to a complete factorization P(z) = a_n(z- α_1)……… (z- α_n), where α_1,……., α_n are not necessarily distinct. Moreover, the factorization is uniquely determined except the order of the factors.

If exactly h of α_j coincide, their common values called a zero of P(z) of order h. Sum of the orders of the zeros of the polynomial is equal to its degree.

Determination of the order of zero :

Suppose α is a zero of P(z) of order h.

$\Rightarrow$ P(z) = $(z-\alpha)^h$ P_h(z) and $P_h(\alpha) \neq 0$ and successive derivative yields,

P(α) = P'(α) =………..= $P^{h-1}(\alpha)$ = 0.

(i.e) the order of a zero equal to order of the first non-vanishing derivative.

NOTE: Zero of order one is called a single zero and characterized by the condition, P(α) = 0 and P'(α) $\neq$ 0.

THEOREM 5: (LUCAS)

If all zeros of a polynomial P(z) lie in a half plane, then all zeros of a derivative P'(z) lie in the same half plane.

Proof: If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$ are zeros of P(z). Then P(z) can be written as, P(z)=a_n(z-α_1)(z-α_2)……..(z-α_n),where $a_n \neq 0$

Taking log on both sides

$$\log P(z) = \log a_n + \log(z-\alpha_1) + \log(z-\alpha_2) + \ldots\ldots + \log(z-\alpha_n)$$

Differentiate with respect to z, we get

$$\frac{\boldsymbol{P'(z)}}{\boldsymbol{P(z)}} = \frac{\mathbf{1}}{\boldsymbol{z-\alpha_1}} + \frac{\mathbf{1}}{\boldsymbol{z-\alpha_2}} + \ldots\ldots + \frac{\mathbf{1}}{\boldsymbol{z-\alpha_n}} \quad \rightarrow \mathbf{(1)}$$

Let the half plane H be defined as the part of the plane where $Im\left(\frac{z-a}{b}\right) < 0$

If α_k is in H and z is not in H

Then we have $Im\left(\frac{z-\alpha_k}{b}\right) = Im\left(\frac{z-a+a-\alpha_k}{b}\right)$

$$= Im\left(\frac{z-a}{b}\right) + Im\left(\frac{-(\alpha_k-a)}{b}\right)$$

$$= Im\left(\frac{z-a}{b}\right) - Im\left(\frac{\alpha_k-a}{b}\right) \quad > 0$$

But the imaginary parts of a reciprocal number have opposite signs.

Therefore, under the same assumption $Im\left(\frac{b}{z-\alpha_k}\right) < 0$

If this is true for all k, therefore from **(1)**

$$b\ \frac{\boldsymbol{P'(z)}}{\boldsymbol{P(z)}} = \sum_{\boldsymbol{k=1}}^{\boldsymbol{n}} \frac{\boldsymbol{b}}{\boldsymbol{z-\alpha_k}}$$

$$Im\left(b\,\frac{P'(z)}{P(z)}\right) = Im\left(\sum_{k=1}^{n} \frac{b}{z-\alpha_k}\right) = \sum_{k=1}^{n} Im\left(\frac{b}{z-\alpha_k}\right) < 0$$

$$P'(z) \neq 0$$

All the zeros of a derivative P’(z) lie in the same half plane H.

RATIONAL FUNCTION

Let $R(z) = \frac{P(z)}{Q(Z)}$ be the quotient of two polynomials. We can assume that P(z) and Q(z) has no common factors and hence no common zero.

R(z) will be given the value of ∞ at the zeros of Q(z). It must therefore must be considered as the function with the values in the extended plane, and as such it is continuous .

The zeros of Q(z) are called poles of R(Z). R'(z)= $\frac{Q(z)P'(z)-P(z)Q'(z)}{Q(z)^2}$ only when Q(z)$\neq$ 0.R'(z) has the same poles as R(z),the order of each poles being increased by 1.

Poles and zeros of a rational function at ∞. $[R(\infty) = \lim_{z\to\infty} R(z)]$

Consider $R_1(z) = R\left(\frac{1}{z}\right)$. (i.e) R(∞) $= R_1(0)$

If $R_1(0) = 0$ or ∞, the order of the zero (or) the pole at ∞ is defined as the order of the zero (or) pole of R_1(z) at the origin

$$\mathrm{R(z)} = \frac{a_0+a_1z+\cdots+a_nz^n}{\mathrm{b_0+b_1z+\cdots+b_mz^m}}$$

We obtain, $R_1(z) = z^{m-n}\,\frac{a_0z^n+a_1z^{\,n-1}+\cdots+a_n}{\mathrm{b_0z^m+b_1z^{m-1}+\cdots+b_m}}$

By the power z^{m-n} belongs either to the numerator or denominator.

Case(i) m > n

⟹ R_1(z) has a zero of order m-n at the origin.

⟹ $R(z)$ has a zero of order m-n at ∞.

Case(ii) m < n

⟹ $R_1(z)$has a pole of order n-m at the origin

⟹ $R(z)$ has a pole of order n-m at ∞

Case(iii)

R(∞) $= R_1(0) = \frac{a_n}{b_m} \neq 0, \infty$

Since R(∞)is neither zero nor ∞

∴ $R(z)$ has neither zero nor a pole at ∞

	Number of zeros			Number of pole		
	In the finite plane	At ∞	Total	In the finite plane	At ∞	Total
$m > n$	n	m - n	m	m	-	m
$m < n$	n	-	n	m	n - m	n
$m = n$	n	-	n	m	-	m

NOTE:

We can now count the total no of zeros and poles in the extended plane. The count shows that the no of zeros including those at ∞ is equal to bigger of m and n.

This common number of zeros and poles is called order of the rational function.

If a is any constant, the function R(z) - a has the same poles as R(z) and consequently the same order .

The zeros of R(z) - a are roots of the equation R(z)=a .

Theorem 6

A rational function R(z) of order p has p zeros and p poles and every equation R(z) = a has exactly p roots.

Proof:

Let $R(z)=\frac{P(z)}{Q(Z)}$ be a rational function.

Consider $R(z) - a = \frac{P(z)}{Q(Z)} - a = \frac{P(z) - aQ(z)}{Q(Z)}$ -----------(1)

The numerator and denominator can not have a common factor.

For, if so, it would be a factor of P(z) and Q(z) both and therefore R(z) would not be in the lowest form. R(z) is not a rational function. This is a contradiction. It follows that the order of R(z) – a = p = order of R(z). Therefore, R(z) – a has exactly p roots.

Theorem 7: Every rational function has a representation by partial fraction:

Proof: First to derive this representation R(z) has a pole at ∞, we carryout the division of P(z) by Q(z) until the degree of the remainder is atmost equal to that of the denominator.

$\therefore R(z) = G(z) + H(z) \longrightarrow$**(1)** Where G(z) is a polynomial without constant term and H(z) is finite at ∞.The degree of G(z) is the order of the pole at ∞ the polynomial G(z) is called the singular part of R(z) at ∞.

Let the distinct finite poles R(z) be denoted by $\beta_1, \beta_{2,\ldots\ldots,}\beta_n$. The function R$\left(\beta_j + \frac{1}{\xi}\right)$ be the rational function of ε with a pole at ξ is equal to ∞.

∴ From decomposition **(1),** R$\left(\beta_j + \frac{1}{\xi}\right) = G_j(\xi) + H_j(\xi)$

Let z = $\beta_j + \frac{1}{\xi}$. Then $R(z) = G_j\left(\frac{1}{z-\beta_j}\right) + H_j\left(\frac{1}{z-\beta_j}\right) \longrightarrow$**(2)**

Here $G_j\left(\frac{1}{z-\beta_j}\right)$ is a polynomial in $\frac{1}{z-\beta_j}$ without constant term called the singular part of R(z) at z = β_j .The function $H_j\left(\frac{1}{z-\beta_j}\right)$ is finite for z = β_j. Consider now the expression, R(z) - G(z) - $\sum_{j=1}^{q} G_j\left(\frac{1}{z-\beta_j}\right)$. $\longrightarrow$**(3)**

This is a rational function which cannot have other poles than $\beta_1, \beta_{2,\ldots\ldots,}\beta_n$ and ∞. At z = β_j, we can find that the two terms with finite limits and the same is true at ∞.

Therefore, **(3)** has neither any finite pole not a pole at ∞. A rational function without poles must reduce to a constant .

If this constant is observed in G(z), We obtain R(z) = G(z)+ $\sum_{j=1}^{q} G_j\left(\frac{1}{z-\beta_j}\right)$

3.3 POWER SERIES

A power series is of the form a_0+a_1z+a_2z^2+……+a_nz^n+…… where the coefficients a_n and the variable z are complex.

NOTE:

$\sum_{n=0}^{\infty} a_n\,(z-\ z_0)^n$ is a power series with respect to the center z_0.consider the geomentric series 1+z+z^2+……..+z^n+…… Whose partial sum is S_n = 1+z+….+z^{n-1} $=\frac{1-z^n}{1-z}$. Since z^n ⟶0 for $|z| < 1$ and $|z^n| \geq 1$ for $|z| \geq 1$,

⟹The geomentric series convergent to $\frac{1}{1-z}$ for $|z| < 1$ and diverges for $|z| \geq 1$.

THEOREM 8 (ABEL)

For every power series $\sum_{n=0}^{\infty} a_n\, z^n$ there exists a number R, $0 \leq R \leq \infty$ called the radius of convergence with the following properties.

1. The series converges absolutely for every z with $|z| < R$. If $0 \leq \rho < R$ the convergence is uniform for $|z| \leq R$.

2. If $|z| > R$ the terms of the series are unbounded and the series is consequently divergent.

3. In $|z| < R$ the sum of the series is an analytic function. The derivative can be obtained by the term wise differentiation and the derivative series has the same radius of convergence.

Proof: The circle $|z| = R$ is called the circle of convergence .We shall show that the theorem holds if we choose R according to the formula

$$\frac{1}{R} = \lim_{n\to\infty} sup|a_n|^{\frac{1}{n}} \longrightarrow \textbf{(1)}$$

This is known as Hadmard's formula.

Let $|z| < R$. Then there exists ρ such that $|z| < \rho < R \implies \frac{1}{\rho} > \frac{1}{R}$

By the definition of limit superior and equation **(1),** there exists a positive integer n_0 such that $|a_n|^{\frac{1}{n}} < \frac{1}{\rho}$ (i.e) $|a_n|^{\frac{1}{n}} < \frac{1}{\rho}$ for all n $\geq n_0 \longrightarrow$**(2)**

$$\implies |a_n z^n| < \left(\frac{|z|}{\rho'}\right)^n \text{ for large n}$$

Since the power series $\sum a_n z^n$ has a convergent geometric series as a majorant and is consequently convergent.

To prove the uniform convergence for $|z| \leq \rho < R$, We choose ρ' with $\rho < \rho' < R$.

From (2), We get $|a_n| < \frac{1}{(\rho')^n}$ for all n $\geq n_0$

$$\implies |a_n z^n| < \left(\frac{|z|}{\rho'}\right)^n < \left(\frac{\rho}{\rho'}\right)^n \text{ for all n} \geq n_0 \quad [as\ |z| \leq \rho]$$

Since the major ant is convergent and has constant terms, we conclude by Weierstrass M-test that the power series is uniformly convergent.

If $|z| > R$, we choose ρ so that $R < \rho < |z|$ $(i.e)\frac{1}{\rho} < \frac{1}{R}$

Since $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} \implies$ There are arbitrary large n such that $|a_n|^{\frac{1}{n}} > \frac{1}{\rho}$

(i.e) $|a_n| > \frac{1}{\rho^n}$ and consequently $|a_n z^n| > \left(\frac{|z|}{\rho}\right)^n$ for infinetly many n. Hence the terms of the series is unbounded accordingly the series is divergent.

STEP:1

The derivative series $\sum_1^\infty n a_n z^{n-1}$ has the same radius of convergent

Proof:

Let R and R' be the radii of convergence of the series $\sum a_n z^n$ and $\sum_1^\infty n a_n z^{n-1}$ respectively.

Then $\frac{1}{R} = \overline{\lim}_{n\to\infty} |\, a_n|^{\frac{1}{n}}$ and $\frac{1}{R'} = \overline{\lim}_{n\to\infty} |\, n a_n|^{\frac{1}{n}}$

$$\frac{1}{R'} = \overline{\lim}_{n\to\infty} n^{\frac{1}{n}} |\, a_n|^{\frac{1}{n}}$$

Therefore the theorem is over if we show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

To prove this, Let $n^{\frac{1}{n}} = 1 + h_n$ so that n $= (1 + h_n)^n$

$$= 1 + n h_n + \frac{n(n-1)}{2} {h_n}^2 + \cdots + {h_n}^n$$

Hence $n > \frac{1}{2} n(n-1){h_n}^2$ **(or)** ${h_n}^2 < \frac{2}{n-1}$ so that $h_n \to 0$ as n$\to \infty$

$Therefore,$ $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ and so R' = R.

STEP 2:

For $|z| < R$ We write $f(z) = \sum_{n=0}^\infty a_n z^n$

$$= s_n(z) + R_n(z)$$

where $s_n(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$

$$R_n(z) = \sum_{k=n}^{\infty} a_k z^k \quad \text{and also} \quad f_1(z) = \sum_1^{\infty} n a_n z^{n-1} = \lim_{n\to\infty} s_n{}'(z)$$

To prove: $f_1(z) = f'(z)$

Consider the identity

$$\frac{f(z)-f(z_0)}{z-z_0} - f(z) = \left(\frac{s_n(z)-s_n(z_o)}{z-z_o} - s_n{}'(z_0)\right) + (s_n{}'(z_0) - f_1(z)) + \left(\frac{R_n(z)-R_n(z_0)}{z-z_0}\right)$$

$\longrightarrow$**(3)** where we assume that $z \neq z_0$ and $|z_0| < \rho < R$

$$\frac{R_n(z)-R_n(z_0)}{z-z_0} = \frac{\sum_{k=n}^{\infty} a_k z^k - \sum_{k=n}^{\infty} a_k z_o{}^k}{z-z_0}$$

$$= \sum_{k=n}^{\infty} \frac{a_k(z^k - z_o{}^k)}{z-z_0}$$

$$= \sum_{k=n}^{\infty} a_k (z^{k-1} + z_0\, z^{k-2} + \cdots + z_0{}^{k-1})$$

$\therefore \left|\frac{R_n(z)-R_n(z_0)}{z-z_0}\right| \leq \sum_{k=n}^{\infty} k|a_k|\ \rho^{k-1}$[since $|z| < \rho$ and $|z_o| < \rho$]

Now $\sum_{k=n}^{\infty} k|a_k|\rho^{k-1}$ is the remainder term in a convergent series .

Hence, we can find n_o such that $\left|\frac{R_n(z)-R_n(z_0)}{z-z_o}\right| <$ €/3 $\forall\, n \geq n_o$ --------(4)

$\lim_{n\to\infty} S'_n(z) = f_1(z)$ for $|z| < R$ and since $|z_o| < R$, $\lim_{n\to\infty} S'_n(z_o) = f_1(z_o)$

=>There exists an n_1 such that $|S_n'(z_o) - f_1(z_o)| <$ €/3 ----------(5) $\forall\, n \geq n_1$.

Choose a fixed $n \geq n_o, n_1$. We know that $S_n'(z_o) = \lim_{z\to z_o} \frac{S_n(z)-S_n(z_o)}{z-z_o}$

By the definition of derivative , we can find $\delta > 0$ such that $0 < |z - z_o| < \delta$

⇨ $\left|\frac{S_n(z)-S_n(z_o)}{z-z_o} - s_n{}'(z_o)\right| <$ €/3 --------(6)

Using (4),(5) and (6) and it follows by (3) that,

$\left|\frac{f(z)-f(z_o)}{z-z_o} - f_1(z_o)\right| <$ € when $0 < |z-z_o| < \delta$

⇨ $f'(z_o)$ exists and $f_1(z_o) = f'(z_o)$.

Remark: Every analytic function has a Taylor development .The power series development of f(z) is uniquely determined if it exists.

A power series with positive radius convergences has derivatives of all orders.

They are given explicitly by

$f(z) = a_o+a_1z+\ldots\ldots\ldots..+a_n z^n +\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$f'(z) = a_1+2a_2z+\ldots\ldots\ldots\ldots\ldots..+na_nz^{n-1}+\ldots\ldots\ldots..$

$f''(z) = 2a_2+6a_3z+\ldots\ldots\ldots\ldots\ldots+n(n-1)a_nz^n+\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$f^k(z) = k!a_k+ \frac{k+1!}{1!} a_{k+1}z + \frac{k+2!}{2!} a_{k+2} z^2 +\ldots\ldots\ldots$

In particular, $a_k = \frac{f^k(0)}{k!}$

∴ The power series becomes $f(z) = f(0) + \frac{f'(0)}{1!} z +\ldots..+ \frac{f^{(n)}(0)}{n!} z^n+\ldots\ldots.$

This is the familiar Maclarian – Taylor development. But we have proved only under the assumption that f(z) has a power series development.

The following theorem refers to the case where a power series converges at a point on the circle of converges at a point on the circle of convergence and note that R = 1.

Theorem : 9 (Abel's Limit theorem)

$\sum a_n$ convergences .Then f(z) = $\sum_{n=0}^{\infty} a_n z^n$ tends to f(1) as z approaches 1. In such a way that $\frac{|1-z|}{1-|z|}$ remains bounded.

Proof.

We may assume that $\sum_{n=0}^{\infty} a_n$=0, since this can be obtained by adding a constant to a_o. Now, f (1) = $\sum_{n=0}^{\infty} a_n$= 0. Let $S_n= a_o+a_1+\ldots\ldots\ldots\ldots+a_n$

Consider the identity (summation by parts)

$S_n(z) = a_o+a_1z+\ldots\ldots\ldots.+a_n z^n = s_o+(s_1-s_o)z+\ldots\ldots\ldots\ldots\ldots+(s_n-s_{n-1})z^n$

$= s_o(1-z) + s_1(z-z^2) +\ldots\ldots\ldots.+ s_{n-1}(z^{n-1}-z^n) + s_n z^n$

$= (1-z)(s_o+ s_1z +\ldots\ldots\ldots\ldots+s_{n-1}z^{n-1}) +s_n z^n$

But $s_n z^n \to 0$ as $n\to \infty$ (since $\sum a_n= 0$, $s_n\to 0$, $z^n \to 0$)

$\Rightarrow$ f (z) = (1-z) $\sum_{n=0}^{\infty} s_n z^n$

Since $\frac{|1-z|}{(1-|z|)}$ remains bounded, there exists a positive constant k such that $\frac{|1-z|}{1-|z|} \leq k.$

Since $S_n\to o \text{ as } n \to \infty$, given $\in > 0$ choose m so large that $|S_n| < \in$ for $n \geq m$

Now $|f(z)| = |(1-z)\sum_0^{\infty} s_n z^n|$

$\leq |(1-z)\sum_{n=0}^{m-1} s_n z^n| + |(1-z)\sum_m^{\infty} s_n z^n|$ --------(1)

and $|(1-z)\sum_m^{\infty} s_n z^n| \leq |1-z| \sum_{n=m}^{\infty} |s_n| |z|^n$

$$< |1-z| \in \sum_m^{\infty} |z|^n$$

$= \left|1-z\right| \in \frac{|z|^m}{(1-|z| \quad)} < \in \frac{|1-z|}{1-|z|} \leq k \in$

Therefore, (1) becomes $\left|f\ (z)\right| \leq \left|1-z\right| \ \left|\sum_0^{m-1} s_n\, z^n\right| + k \in$

The first term on the right can be made arbitary small by choosing z sufficiently close to1.

Therefore, $f(z) \to 0 = f(1), as\ z \to 1$ subject to the stated restriction.

PROBLEM 1:

1).Find the radius convergences of the following series.

i) $\sum n^p z^n$ ii)$\sum \frac{z^n}{n!}$ iii)$\sum n! z^n$ iv)$\sum (1+1/n)^{n2} z^n$

<u>soln :</u>

i) $a_n = n^p$ $a_{n+1} = (n+1)^p$

$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{n^p}{(n+1)^p} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p$

$=> \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^p} = 1$

ii) $\sum \frac{z^n}{n!}$

$a_n = \frac{1}{n!}$ $a_{n+1} = \frac{1}{n+1!}$

$R = \lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} \frac{(n+1)n!}{n!} = \lim_{n\to\infty} (n+1) = \infty$

iii) For z = 0 the series is convergent.

$a_n = n!$, $a_{n+1} = (n+1)!$.

Then R= $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{n!(n+1)} = \lim_{n\to\infty} \frac{1}{n+1} = 0$

iv) $a_n = (1+\frac{1}{n})^n$

$\frac{1}{R} = \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} (1+\frac{1}{n})^n = e$. Therefore, $R = \frac{1}{e}$.

Problem 2. If $\sum a_n z^n$ and $\sum b_n z^n$ have the radii of convergences R_1 and R_2 .Show that the radius of convergences for $\sum a_n b_n z^n$ is atleast R_1R_2 .

Soln: $\frac{1}{R_1} = \overline{\lim_{n\to\infty}} |a_n|^{1/n}$ and $\frac{1}{R_2} = \overline{\lim_{n\to\infty}} |b_n|^{1/n}$

Let R be the radius of convergence of $\sum a_n b_n z^n$

$\frac{1}{R} = \overline{\lim_{n\to\infty}} |a_n b_n|^{1/n} = \overline{\lim_{n\to\infty}} |a_n|^{1/n} |b_n|^{1/n} = \overline{\lim_{n\to\infty}} |a_n|^{1/n} \overline{\lim_{n\to\infty}} |b_n|^{1/n} = \frac{1}{R_1} . \frac{1}{R_2}$

=> $R=R_1R_2$

Problem3. Find the radius of convergences of a power series f (z) $=\sum_0^\infty \frac{z^n}{2^n+1}$ and prove that (2-z) f (z) - 2→ $0 \; as \; z \to 2$

Solution: $a_n = \frac{1}{2^n+1}$, $a_{n+1} = \frac{1}{2^{n+1}+1}$. and $R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$

$$R = \lim_{n\to\infty} \frac{2^{n+1}+1}{2^n+1} = \lim_{n\to\infty} \frac{2^n(2+\frac{1}{2^n})}{2^n(1+\frac{1}{2^n})} = \frac{2+0}{1+0} = 2$$

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^n+1} < \sum_{n=0}^{\infty} \frac{z^n}{2^n} = 1+z/2+z^2/2^2+\ldots\ldots\ldots$$

$$=> \frac{1}{1-z/2} = \frac{2}{2-z} \; [\text{ since } |z| < 2 \; \forall z]$$

$$\lim_{z\to 2} (2-z) f(z) = \lim_{z\to 2} (2-z) \frac{2}{2-z} = 2$$

Therefore, $(2-z)f(z) - 2 \to 0 \; as \; z \to 2.$

UNIT IV

COMPLEX INTEGRATION

4.1 The Line Integrals :

Definite integral of complex function over a real interval . If f (t) = u(t)+i v(t) is a continuous function defined in an interval (a,b) .Then define,

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

Properties of the integral

Property 1 : $\int_a^b cf(t)dt = c\int_a^b f(t)dt$

Proof: Let c $=\propto + i\beta$ and f(t)= u(t)+i v(t).

$$\int_a^b c\, f(t)dt = \int_a^b (\propto + i\beta)\big(u(t) + i\, v(t)\big)\ dt$$

$$=\int_a^b [(\propto u - \beta v) + i\ (\alpha v + \beta u)]\ dt$$

$$=\int_a^b (\propto u - \beta v)dt + i\int_a^b (\alpha v + \beta u)dt \ldots\ldots\ldots(1) \qquad \text{(by defn)}$$

$c\int_a^b f(t)$ dt = $(\propto + i\beta)\int_a^b (u(t) + iv(t)\ )dt$

$$=\propto \int_a^b u(t)dt\ + i\ \alpha\int_a^b v(t)dt + i\beta\int_a^b u(t)dt - \beta\int_a^b v(t)dt$$

$$=\int_a^b \propto u\ dt + i\int_a^b \propto v\ dt + i\int_a^b \beta u\ dt - \int_a^b \beta v\ d\,t$$

$$=\int_a^b (\propto u - \beta v)\ dt + i\int_a^b (\alpha v + \beta u)\ dt \quad \ldots\ldots\ldots(2)$$

From (1) and (2), $\int_a^b c\ f(t)dt = c\int_a^b f(t)dt$

Property 2: When $a \leq b$, $\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|\ dt$ holds for arbitrary complex function f(t).

Proof:

If $\int_a^b f(t)dt = 0$ then $\int_a^b f(t)dt \geq 0 = \int_a^b f(t)dt$

Clearly, $\left| \int_a^b f(t)dt \right| \leq \int_a^b \left| f(t) \right| dt$

Therefore, the given statement is true .

Now assume that $\int_a^b f(t)dt \neq 0$.

From (1), $\mathrm{Re}\left[\int_a^b c\, f(t)dt\right] = \mathrm{Re}\left[c \int_a^b f(t)dt\right]$

Since 'c' is arbitrary, we may set c = $e^{-i\theta}$ where θ is real but arbitrary.

$\mathrm{Re}\left[e^{-i\theta} \int_a^b f(t)dt\right] = \mathrm{Re}\left[\int_a^b e^{-i\theta} f(t)dt\right]$

$= \int_a^b Re\ [\, e^{-i\theta} f(t)\,]\ \mathrm{dt} \quad \leq \int_a^b \left| e^{-i\theta} f(t) \right| dt$

$\leq \int_a^b \left| f(t) \right| dt$ …………….(1)

Since $\theta\ is\ arbitrary\ we\ may\ set\ \theta = \arg(\int_a^b f(t))\ dt$

Then $\int_a^b f(t)\ dt = \left| \int_a^b f(t)\ dt \right| e^{i\theta}$………..(2)

$\mathrm{Re}\left[e^{-i\theta} \int_a^b f(t)dt\right] = \mathrm{Re}\left[e^{-i\theta} \left| \int_a^b f(t)dt \right| e^{i\theta}\right]$

$= \mathrm{Re} \left| \int_a^b f(t)dt \right|$

$= \left| \int_a^b f(t)dt \right|$……….(3)

From (1) and (3), We have $\left| \int_a^b f(t)dt \right| \leq \int_a^b \left| f(t) \right| dt.$

Complex line integral of f(z) extended over the arc γ

Suppose γ is a smooth arc is given by z = z (t), a ≤ t ≤ b and f (z) is continuous on γ . Then f (z (t)) is continuous in t. We define

$$\int_\gamma f(z)\, dz = \int_a^b f\big(z(t)\big) z'(t)\, dt.$$

If γ is piecewise differentiable or if z'(t) is piecewise continuous the interval can be subdivided in the obvious manner.

The integral is invariant under change of parameter. A change of parameter is defined by increasing function t = t(τ) which maps an interval α ≤ τ ≤ β into a ≤ t ≤ b. we assume that t (τ) is piecewise differentiable.

By change of variables,

$$\int_a^b f\big(z(t)\big) z'(t)\, dt = \int_\alpha^\beta f\ \Big(z\big(t(\tau)\big)\Big)\, z'\big(t(\tau)\big) t'(\tau) d\tau$$

But $z'\big(t(\tau)\big) t'(\tau) = \frac{d}{d\tau}$ (z (t(τ))).

$\therefore \int_a^b f\big(z(t)\big) z'(t)\, dt = \int_\alpha^\beta f\ \Big(z\big(t(z)\big)\Big) \frac{d}{dz}$ (z (t(z))) dz.

Hence the integral has the same value whether γ is represented by z = z (t) or by

z = z (t (τ)).

Note: We define the opposite are – γ by the equation z = z (-t), -b ≤ t ≤ -a.

$\therefore \int_{-\gamma} f(z)\, dz = \int_{-b}^{-a} f\big(z(-t)\big)\, d(z(-t))$

$= - \int_a^b f\big(z(s)\big) d\big(z(s)\big)$ by the change of variable, z = - t

$= -\int_\gamma f(z)\,dz$

Further if $\gamma = \gamma_1 + \gamma_2 + \ldots\ldots + \gamma_n$ then

$$\int_\gamma f(z)\,dz = \int_{\gamma_1+\gamma_2+\cdots+\gamma_n} f(z)dz =$$

$$= \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \ldots\ldots\ldots + \int_{\gamma_n} f(z)dz$$

Integration with respect to arc length

$$\int_\gamma f(z) \mid dz \mid = \int_a^b f\big(z(t)\big) \mid z'(t)dt \mid = \int_a^b f\big(z(t)\big) \mid z'(t) \mid dt$$

$$\left|\int_\gamma f(z)dz\right| = \left| \int_a^b f\big(z(t)\big) z'(t)\,dt \right| \leq \int_a^b \mid f(z(t)) \mid |z'(t)|dt$$

$$= \int_\gamma \mid f(z) \mid \mid dz \mid$$

Note : If f = 1 then $\left|\int_\gamma dz\right| \leq \int_\gamma \mid dz \mid = length\ of\ \gamma$

$$\int_\gamma f\,ds = \int_\gamma f \mid dz \mid = \int_a^b f\big(z(t)\big) \mid z'(t) \mid dt \quad \text{(as } ds = dx^2+dy^2\text{)}$$

To find the length of the circle with radius ρ with centre at a:

The parametric equation of circle is z = z(t) = a+ρe^{it}, $0 \leq t < 2\pi$. $z'(t) = \rho i e^{it}$.

$$\int_\gamma ds = \int_\gamma \mid dz \mid = \int_0^{2\pi} \mid z'(t) \mid dt| \quad = \int_0^{2\pi} \rho dt \quad = \rho\, 2\pi$$

RECTIFIABLE ARCS

The length of an arc can also be defined as the least upper bound of all sums.

$|z(t_1) - z(t_0)| + |z(t_2) - z(t_1)| \ldots\ldots\ldots\ldots\ldots + |z(t_n) - z(t_{n-1})|$

where $a = t_0 < t_1 < \ldots\ldots < t_n = b$

If the l.u.b is finite we say that the arc is rectifiable.

Note : Piecewise differentiable arcs are rectifiable arcs.

Observation: An are z = z (t) is rectifiable iff real and imaginary part of z (t) are of bounded variation.

For, Since $|x(t_k)-x(t_{k-1})| \leq |z(t_k)-z(t_{k-1})|$ and

$$|y(t_k)-y(t_{k-1})| \leq |z(t_k)-z(t_{k-1})|$$

$$|z(t_k)-z(t_{k-1})|=|z(t_k)+iy(t_k)-(x(t_{k-1})+iy(t_{k-1})|$$

$$=|[x(t_k)-x(t_{k-1})]+i[y(t_k)-y(t_{k-1})]|$$

$$\leq |x(t_k)-x(t_{k-1})| + |y(t_k)-y(t_{k-1})|$$

The sums $|z(t_1)-z(t_0)|+|z(t_2)-z(t_1)|+\ldots\ldots\ldots\ldots+|z(t_n)-z(t_{n-1})|$ and

the sums $|x(t_1)-x(t_0)|+|x(t_2)+x(t_1)|+\ldots\ldots\ldots\ldots+|x(t_n)-x(t_{n-1})|$,

$|y(t_1)-y(t_0)|+|y(t_2)-y(t_1)|+\ldots\ldots\ldots\ldots+|y(t_n)-y(t_{n-1})|$ are bounded at the same time.

When the later sums are bounded, one says that the functions x(t) and y(t) are of bounded variation.

Therefore, a arc z = z (t) is rectifiable iff real and imaginary part of z (t) are of bounded variation.

LINE INTEGRALS AS FUNCTION OF ARC:

General line integrals of the form $\int_\gamma pdx+qdy$ are often studied as function or functional of the arc γ.

Assume that $p\ and\ q\ are\ defined\ and\ continuous$ in a region $\Omega\ and\ \gamma$ is free to vary on Ω.

Class of integrals having the property that the integral over an arc depends on its end points.

If γ_1 and γ_2 have the same initial point and the same end points then $\int_{\gamma_1} pdx + qdy = \int_{\gamma_2} pdx + qdy$.

Theorem 1:

Integrals depend only on the end points iff the integral over any closed curve is zero.

Proof:

If γ is a closed curve then γ and $-\gamma$ have the same end points and if the integrals depend only on the end points.

⇨ $\int_{\gamma} = \int_{-\gamma} = -\int_{\gamma}$

⇨ $2\int_{\gamma} = 0$. $\int_{\gamma} = 0$

Conversely, Suppose γ_1 and γ_2 have the same end points.

(i.e.) T.P the integral depends only on the end points.

(i.e.) To Prove: $\int_{\gamma 1} = \int_{\gamma 2} where\, \gamma_1\, and\, \gamma_2$ have the same end points.

By given hypothesis, $\gamma_1 - \gamma_2$ is a closed curve

Since the integral over any closed curve is zero.

⇨ $\int_{\gamma_1-\gamma_2} = 0 \Rightarrow \int_{\gamma_1} + \int_{-\gamma_2} = 0 \Rightarrow \int_{\gamma_1} - \int_{\gamma_2} = 0$

⇨ $\int_{\gamma_1} = \int_{\gamma_2}$

The following theorem gives the necessary and sufficient condition for a line integral depends only on the end points

THEOREM 2:

The line integral $\int_\gamma pdx + q\,dy$ defined in Ω, depends only on the end points of γ iff $there\ exists\ a\ function$ U(x,y) in Ω with the partial derivatives $\frac{\partial U}{\partial x} = p,\ \ \frac{\partial U}{\partial y} = q.$

Proof:

Suppose there exists a function U(x,y) in Ω such that $\frac{\partial U}{\partial x} = p,\ \frac{\partial U}{\partial y} = q.$

To Prove the integral depends only on the end points of γ

Let a, b are the end points of γ

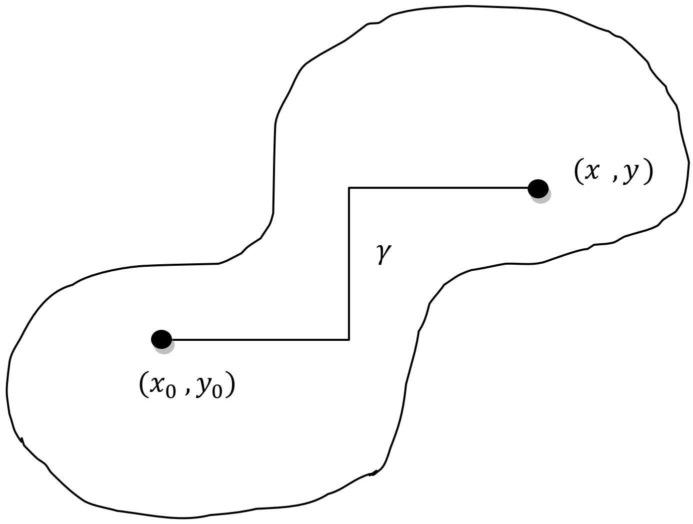

FIG. 4.1

Then $\int_\gamma pdx + qdy = \int_\gamma \frac{\partial U}{\partial x}\ dx + \frac{\partial U}{\partial y}\ \ dy$

$$= \int_a^b \frac{\partial U}{\partial x}\ x'(t)dt + \frac{\partial U}{\partial y} y'(t)dt$$

$$= \int_a^b \frac{d}{dt}\ (U\big(x(t), y(t)\big))\ dt$$

$= \int_a^b d\,(U(x(t), y(t)))$

$= [U(x, (t), y\ (t))]\,_a^b$

= U(x (b),y (b)) – U (x (a), y (a))

∴ The line integral $\int_\gamma pdx + q\ dy$ depends only on the end points of γ.

Conversely, Suppose the line integral $\int_\gamma pdx + q\ dy$ depends only on the end points of γ.

To Prove: There exists a function U(x,y) in Ω such that $\frac{\partial U}{\partial x} = p, \frac{\partial U}{\partial y} = q.$

Choose a fixed point $(x_{0,}y_0) \in \Omega$ joint it to (x, y) by a polygon γ contained in Ω whose sides are parallel to the co-ordinate axes.

Define a function U(x, y) = $\int_\gamma pdx + q\ dy$

Since the line integral depends only on the end points of γ, the function is well defined. Choose the last segment of γ horizontal, we can keep y constant and let x vary without changing the other segment.

On the last segment we can choose x for parameter and obtained

U(x,y) = $\int^x P\ (x, y)dx + constant$

The lower limit of the integral being irrelevant , $then\ \ \frac{\partial U}{\partial x} = p.$

In the same way, by choosing the last segment vertical by keeping x constant,

We have U(x,y) = $\int^x q\ (x, y)dx + constant$

⇨ $\frac{\partial U}{\partial y} = q$

Note:

If $pdx + qdy = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial x}\, dy = dU,$ then $pdx + qdy$ is an exact differential.

Therefore, the above theorem can be stated as, An integral depends only on the end points if and only if the integrand is an exact differential.

Consider f (z) dz = f (z) dx + i f (z) dy

By the definition of an exact differential, there must exist a function F(z) in Ω such that $\frac{\partial F(z)}{\partial x} = f(z), \frac{\partial\, F(z)}{\partial y} = i\, f(z).$

$\therefore \frac{\partial F}{\partial x} = -i\, \frac{\partial F}{\partial y}$ which is the complex form of C – R equations.

Further, f (z) by assumption is continuous. (Otherwise $\int_\gamma f(z)dz$ is not defined)

Hence F(z) is analytic.

∴The above theorem can be restated as follows

The integral $\int_\gamma f(z)dz$ with continuous f depends only on the end points of γ iff f is the derivative of an analytical function in Ω.

Lemma 3:

We find that $\int_\gamma (z-a)^n$ dz = 0 for all closed curve γ provided that the integer n ≥ 0.

Proof:

For, since f (z) = $(z\text{-}a)^n$ is continuous and f is the derivative of an analytic function

F (z) $=\frac{(z-a)^{n+1}}{(n+1)}$ in the whole plane.

$\therefore$By the result above, $\int_\gamma f(z)dz$ depends only on the end points of γ.

⇨ the integral over any closed curve is zero.

$\therefore \int_\gamma (z-a)^n$ dz = 0 for any closed curve γ. …….(1)

If n is negative but not equal to -1.

The same result hold for all closed curves which do not pass through a. In the complementary region of the point a the indefinite integral is still analytic.

For n = -1, the equation (1) does not always hold.

Consider an example.

Consider a circle C with centre a represented by the equation z = a +ρ e^{it}, $0 \leq t \leq 2\pi$. Then dz = $\rho\, ie^{it}\, dt$.

We obtain $\int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{\rho\, i\, e^{\,it}}{\rho e^{it}}dt = 2\pi i \neq 0$

Example 1. Compute $\int_\gamma xdz$ where γ the directed line segment from 0 to 1+ i.

Soln: Let z = x+iy , z = 0 , ⇒ x = 0, y = 0 and z = 1+i ⇒ x=1, y=1

Therefore, y = x ⇒ dx = dy

$dz = dx + idy \quad = dx + idx \quad = (1+i)dx$

Therefore, $\int_\gamma xdz = \int_0^1 x(1+i)dx = (1+i)(\frac{x^2}{2})\Big|_0^1 = \frac{1+i}{2}$

Example 2: Compute $\int_{|z|=1} |\, z-1 \,||\, dz \,|$

Soln: Given |z|=1 then z = |z|e^{it} = e^{it} $=$ cos t $+$ i sin t, $0 \leq t \leq 2\pi$

$$| z-1 |^2 = | \cos t + i\, sint - 1 |^2 = 2(1-cost) \quad = 4sin^2\frac{t}{2}$$

Now, dz = ie^{it} dt implies |dz| = dt. Then,

$$\int_{|z|=1} | z-1 || dz | \quad = \int_0^{2\pi} 2sin\frac{t}{2}\,\text{dt} = 2\left[\frac{-cos\frac{t}{2}}{\frac{1}{2}}\right]_0^{2\pi} = \text{-4}(\cos\pi \text{ - cos0}) = 8.$$

4.2 Cauchy's Theorem for a rectangle

Consider a rectangle R defined by the inequality a ≤ x ≤ b and c ≤ y ≤ d. This perimeter can be considered as a simple closed curve consisting of four line segments whose direction we choose so that R lies to the left of the directed segment. The order of the vertices is (a,c), (b,c), (b,d), (a,d). We refer to this closed curve as the boundary curve (or) contour of arc and we denote it by ∂R.

R is chosen as a closed point set and hence it is a region. Further, a function is analytic on the rectangle R means that it is analytic on an open set which is contained in R.

Theorem 4:

The function f(z) is analytic on R. Then $\int_{\partial R} f(z)dz$ = 0.

Proof: Proof is based on the method of bisection. Define $\eta(R) = \int_{\partial R} f(z)dz$,

which we will use for any rectangle contained in the given one.

If R is divided into 4 congruent rectangles $R^{(1)}, R^{(2)}, R^{(3)}$, $R^{(4)}$ by joining the mid points of opposite sides.

We denote the boundaries of the rectangles $R^{(k)} as \;\; \partial R^{(k)}$, k = 1, 2, 3, 4.

Therefore, We find that $\partial R = \partial R^{(1)} + \partial R^{(2)} + \partial R^{(3)} + \partial R^{(4)}$, since the common sides cancel each other.

Now,$\int_{\partial R} f(z)dz = \int_{\partial R^{(1)}} f(z)dz + \int_{\partial R^{(2)}} f(z)dz + \int_{\partial R^{(3)}} + \int_{\partial R^{(4)}} f(z)dz$

Denote $\int_{\partial R^{(k)}} f(z)dz = \eta\left(R^{(k)}\right)$

Clearly for at least one R^k (k = 1, 2, 3, 4), we have $|\,\eta(R^{(k)})\,| \geq \frac{1}{4}\,\eta\,(R)$.

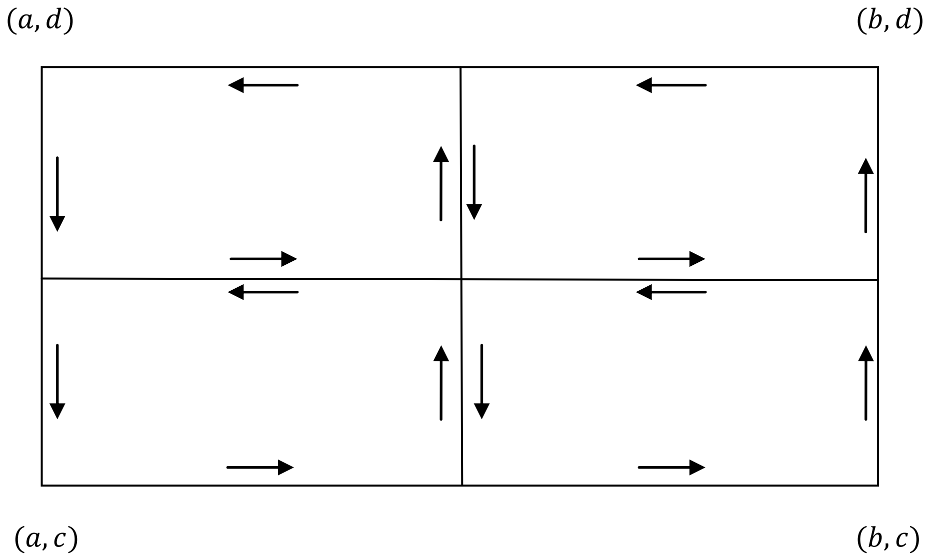

FIG. 4-2. Bisection of rectangle.

We denote this rectangle by $R^{(1)}$. If several R^k have this property that choice shall be made according to the definite rule. This process can be repeated indefinitely and be obtain the sequence of nested rectangle or $R \supset R_1 \supset R_2 \supset \ldots\ldots \supset R_n \supset \ldots\ldots$

With the properties, $|\eta\,(R_n\,)\,| \geq \frac{1}{4}\,|\eta(R_{n-1})\,|$

$$\geq \frac{1}{4}\,(\frac{1}{4}|\,\eta(R_{n-1})\,|) = \frac{1}{4^2}\,|\,\eta(R_{n-2})\,| \quad .$$

$$\geq \frac{1}{4^n}\,|\,\eta(R)\,|$$

Therefore, $|\eta\,(R_n)\,| \geq \frac{1}{4^n}\,|\,\eta(R)\,| \dashrightarrow (\,1\,)$

The rectangle R_n converges to a point $z^* \in R$ in the sense that R_n will be contained in the prescribed neighborhood $|z-z^*| < \delta$ as soon as n sufficiently large.

First of all, we choose δ so small that f (z) is defined and analytic in $|z-z^*| < \delta$.

Secondly, if $\in > 0$ is given, we choose δ so that

$$|z-z^*| < \delta \Rightarrow \left| \frac{f(z)-f(z*)}{z-z*} - f'(z*) \right| < \in$$

$$\Rightarrow | f(z) - f(z^*) - f'(z^*)(z-z^*) | < \in | z - z^* | \ for \ | z - z^* | < \delta \longrightarrow (2)$$

We assume that δ satisfies both conditions R_n is contained in $|z-z^*| < \delta$.

Also we have $\int_{\partial R_n} 1\, dz = 0$, $\int_{\partial R_n} z\, dz = 0$ [$\therefore$ 1 and z are the derivative of analytic function z and $\frac{z^2}{2}$respectively]

$\Rightarrow$ $\eta(R_n) = \int_{\partial R_n} f(z)dz$

$$= \int_{\partial R_n} f(z)dz - f(z^*)\int_{\partial R_n} dz - f'(z^*)\int_{\partial R_n} z\, dz + f'(z^*)\ z^*\int_{\partial R_n} dz$$

$$= \int_{\partial R_n} [\big(f(z) - f(z^*)\big) - f'(z^*)\,(z-z^*)]dz$$

Therefore, $$|\eta(R_n)| \leq \int_{\partial R_n} |f(z) - f(z^*) - (z-z^*)f'(z^*)|\,|dz|$$

$$\leq \int_{\partial R_n} \in |z-z^*|\,|dz|$$

If d_n denotes the length of the diagonal of R_n then $z \in R_n$ and so $| z - z^* | \leq d_n$

If L_n denotes the length of the perimeter of R_n then

$|\eta(R_n)| \leq \in d_n \int_{\partial R_n} |dz| = \in d_n L_n.$

If d and L denotes the length of the diagonal and the perimeter of R respectively, then $d_n = 2^{-n}$ d and $L_n = 2^{-n}$ L. Therefore, $|\eta(R_n)| \leq 4^{-n}\, d\, L \in \longrightarrow (3)$

From (1) and (3) , $|\eta(R_n)| \geq \frac{1}{4^n} |\eta(R)|$

$$|\eta(R)| \leq 4^n |\eta(R_n)| \leq 4^n 4^{-n} \; d\, L \in \leq d\, L \in$$

Since $\in$ is arbitrary, we have $\eta(R) = 0$. Hence $\int_{\partial R} f(z)dz = 0$.

Theorem 5

Let f(z) be analytic on the set R' obtained from a rectangle R by omitting a finite number of interior points ξ_i. If it is true that $\lim_{z \to \xi_i}(z - \xi_i)f(z) = 0$. for all I, then $\int_{\partial R} f(z\,)dz$

Proof: It is sufficient consider the case of a single exceptional point ξ. (For evidently R can be divided into smaller rectangle which contains at most one ξ_i)

We divided R into 9 rectangles as shown in the figure.

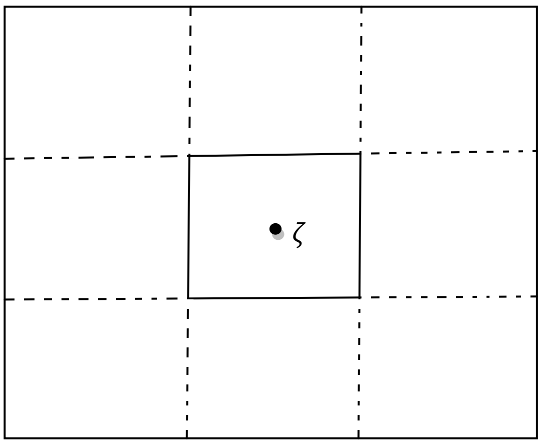

FIG. 4-3

Therefore, $\int_{\partial R} f(z)dz = \sum_{i=0}^{8} \int_{\partial R_i} f(z)dz$.

Hence, by Cauchy theorem for rectangle applied to all rectangles except R_0 $\int_{\partial R_i} f(z)dz = 0$, i = 1, ….,8.

Therefore, $\int_{\partial R} f(z)dz = \int_{\partial R_0} f(z)dz$. Given, $\lim_{z \to \xi}(z - \xi)f(z) = 0$.

If $\epsilon > 0$ we can choose the rectangle R_0 so small that $|z - \xi|\,|f(z)| < \epsilon$.

Consider $|f(z)| < \frac{\epsilon}{|z-\xi|}$ on ∂R_0.

$$\left|\int_{\partial R_0} f(z)dz\right| \leq \int_{\partial R_0} |f(z)||dz| < \epsilon \int_{\partial R_0} \frac{|dz|}{|z-\xi|} \quad \text{....(1)}$$

Let us assume R_0 is a square with centre ξ and a be the side of the square R_0.

Therefore, $|z - \xi| \geq \frac{a}{2} \Rightarrow \frac{1}{|z-\xi|} \leq \frac{2}{a}$.

Now, $\int_{\partial R_0} \frac{|dz|}{|z-\xi|} \leq \frac{2}{a} \int_{\partial R_0} |dz| = \frac{2}{a}\, 4a = 8$.

Therefore (1) becomes, $\left|\int_{\partial R_0} f(z)dz\right| < \epsilon$. Since ϵ is arbitrary, the theorem follows.

Note: The hypothesis of the theorem is fulfilled if f(z) is analytic and bounded on R'.

Cauchy's theorem in a disc

It is not proved that integral of an analytic function over a closed curve is always is zero. $\int_C \frac{dz}{z-a} = 2\pi i \textit{ where } C \textit{ is a circle.}$

In order to make sure that the integral vanishes, it is necessary to make a special assumption concerning the region Ω in which f(z) is known to be analytic and to which the curve γ is restricted.

We must restrict to a special case we assume that Ω is a open disc $|z - z_0| < \rho$ to be denoted by Δ.

Theorem 6:

If f(z) is analytic in a open disc Δ, then $\int_\gamma f(z)dz = 0$ for every closed curve γ in Δ.

Proof: Let O be centre $z_0 = x_0 + i\, y_0$ and P be any point z = x + iy inside Δ. We define a function F(z) = $\int_\sigma f(z)dz$……..(1) where σ consists of the horizontal line segment OA from the centre (x_0, y_0) to (x, y_0) and the vertical segment AP from (x, y_0) to (x, y).

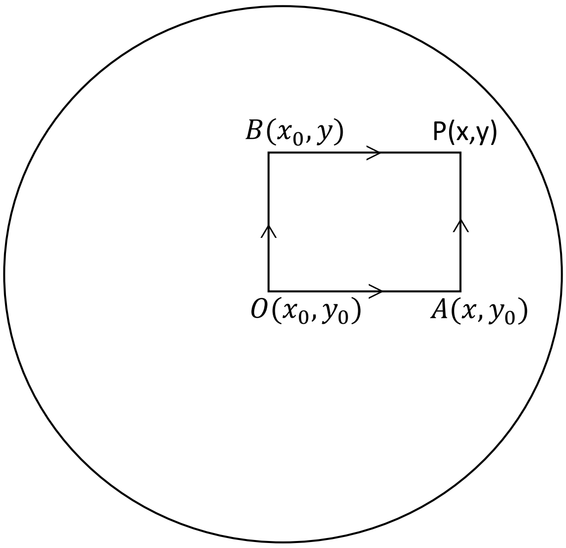

FIG. 4-4

F(z) = $\int_{OAP} f(z)dz$ = $\int_{OA} f(z)dz + \int_{AP} f(z)dz \ldots\ldots(2)$

$$= \int_{x_0}^{x} f(t + iy_0)dt + \int_{y_0}^{y} f(x + it)dt \ldots\ldots(3)$$

From the figure, OAPBO is a rectangle. By Cauchy theorem of rectangle, $\int_{OAPBO} f(z)dz = 0$.

Let σ_1 be a curve consists of the vertical segment OB from (x_0, y_0) to (x_0, y) and the horizontal segment BP from (x_0, y) to (x, y).

OAPBO = OAP + PBO = $\sigma + (-\sigma_1) = \sigma - \sigma_1$. Therefore, $\int_{\sigma - \sigma 1} f(z)dz = 0$.

$\Longrightarrow \int_{\sigma} f(z)dz + \int_{-\sigma 1} f(z)dz = 0 \Longrightarrow \int_{\sigma} f(z)dz - \int_{\sigma 1} f(z)dz = 0$

$\Longrightarrow \int_{\sigma} f(z)dz = \int_{\sigma 1} f(z)dz$

Therefore, $F(z) = \int_{\sigma 1} f(z)dz = \int_{OB} f(z)dz + \int_{BP} f(z)dz \ldots\ldots..(4)$

$= \int_{y_0}^{y} f(x_0 + it)idt + \int_{x_0}^{x} f(t + iy)dt$

$= i\int_{y_0}^{y} f(x_0 + it)dt + \int_{x_0}^{x} f(t + iy)dt \ldots..(5)$

From (3), $\frac{\partial F}{\partial y} = i\, f(x+iy) = i\, f(z)$

From (5), $\frac{\partial F}{\partial x} = f(x+iy) = f(z) \Longrightarrow \frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y} = f(z) - f(z) = 0$

If $F(z) = u + iv$ then $\frac{\partial(u+iv)}{\partial x} + i\frac{\partial(u+iv)}{\partial y} = 0$

$\Longrightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = 0 \Longrightarrow (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) + i(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) = 0$

$\Longrightarrow \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$ and $\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \Longrightarrow u_x = v_y$ and $v_x = - u_y$

$\Longrightarrow u_x = v_y$ and $u_y = -v_x$

$\Longrightarrow$ u and v satisfy C.R equations.

Now, $\frac{\partial F}{\partial x} = f(z)$, $\frac{\partial F}{\partial y} = i\, f(z)$ and f(z) is continuous

$\Longrightarrow u_x, u_y, v_x, v_y$ are all continuous. Therefore, $F(z) = u + iv$ is analytic on Δ

$\Longrightarrow$ f(z) dz is exact differential

$\Rightarrow$ The integral depends only on the endpoints.

$\Rightarrow$ The integral over any closed curve γ in Δ is zero.

Therefore, $\int_\gamma f(z)dz = 0$.

Theorem 7:

Let f(z) be defined in the region Δ' obtained by omitting a finite number of points ξ_i from an open disc Δ. If f(z) satisfies the condition $\lim_{z \to \xi_i (} z - \xi_i) f(z) = 0$ for all i then $\int_\gamma f(z\,)dz$ holds for any closed curve γ in Δ'.

Proof: Let O be the centre $z_0 = x_0 + i\, y_0$ and and P be any point inside Δ. We define F(z) by F(z) = $\int_\sigma f(z)dz$ where we let the curve σ not passing exceptional points. Assuming first that no ξ_i lies on the line $x = x_0$ and $y = y_0$ by letting σ consists of three line segments as in the figure with the last segment is vertical and consider σ' with the last segment is horizontal (F(z) is independent of the choice of the middle segment)

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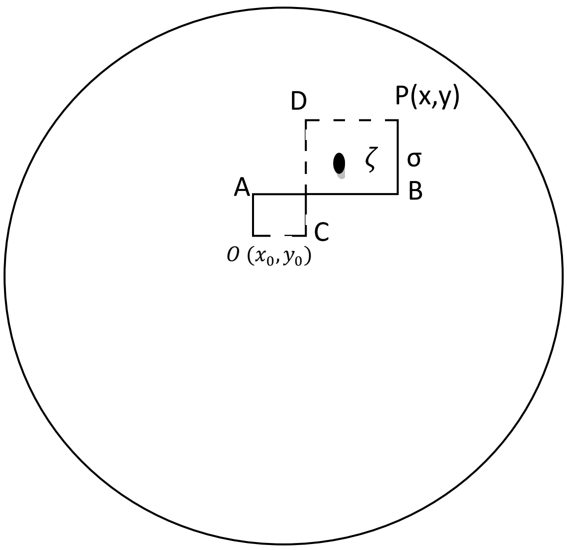

FIG. 4-5

By Theorem 4 and Theorem 5 , $\int_{\mathrm{OABPDCO}} f(z)dz = 0$.

$\Rightarrow \int_{\mathrm{OABP}} f(z)dz - \int_{\mathrm{OCDP}} f(z)dz = 0$

$\Rightarrow \int_{\mathrm{OABP}} f(z)dz = \int_{\mathrm{OCDP}} f(z)dz$

$\Rightarrow F(z) = \int_{\sigma} f(z)dz = \int_{\sigma'} f(z)dz$

It is easy to verify that $\frac{\partial F}{\partial y} = if(z)$, $\frac{\partial F}{\partial x} = f(z)$. Hence $\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$

$\Rightarrow$ F is analytic $\Rightarrow f(z)dz$ is exact differential

$\Rightarrow \int_{\gamma} f(z)dz = 0$ for any closed curve γ.

4.3 Cauchy's Integral Formula

It enables us to study the local property of an analytic function.

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Lemma 8: (The index of a point with respect to a closed curve)

If the piecewise differentiable closed curve γ does not pass through the point a, then the value of the integral $\int_{\gamma} \frac{dz}{z-a}$ is a multiple of $2\pi i$.

Proof

The equation of γ is $z = z(t), \alpha \leq t \leq \beta$

Let us consider the function $h(t) = \int_{\alpha}^{t} \frac{z'(t)}{z(t)-a}\, dt$

It is defined and continuous on the closed interval $[\alpha, \beta]$ and it has the derivative

$h'(t) = \frac{z'(t)}{z(t)-a}$ ____________(1) when ever $z'(t)$ is continuous

Consider, $\frac{d}{dt}\left[e^{-h(t)}(z(t)-a)\right]$

$= e^{-h(t)}z'(t) - (z(t)-a)e^{-h(t)}h'(t)$

$= e^{-h(t)}z'(t) - (z(t)-a)e^{-h(t)}\frac{z'(t)}{z(t)-a} = 0$ except perhaps at a finite number of points , (using (1)).

$e^{-h(t)}(z(t) - a)$ = a constant = k (say)…….(2)

When t = α, h(α) = 0 .Therefore, $e^{-h(\alpha)}(z(\alpha) - a)$ = k $\Longrightarrow (z(\alpha) - a) = k.$

Therefore, (2) becomes $e^{-h(\alpha)} = \frac{(z(\alpha)-a)}{(z(t)-a)} \Longrightarrow e^{h(t)} = \frac{(z(t)-a)}{(z(\alpha)-a)}$.

When t = β , $e^{h(\beta)} = \frac{(z(\beta)-a)}{(z(\alpha)-a)}$. Since γ is a closed curve, z(α) = z(β).

Therefore, $e^{h(\beta)} = 1. \Longrightarrow$ h(β) is a multiple of 2πi.

Hence, h(β) = $\int_{\alpha}^{\beta} \frac{z'(t)dt}{z(t)- a}$ = multiple of 2πi.

Therefore, $\int_{\gamma} \frac{dz}{z-a}$ = h(β) = multiple of 2πi.

Definition: (The index or winding number)

The index of the point a with respect to the curve γ by the equation

$\eta(\gamma$,a) = $\frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z-a}$. It is also called the winding number of γ with respect to a.

Properties of winding number:

Property 1. Prove that $\eta(-\gamma$,a) = - $\eta(\gamma$,a)

Proof:

$$\eta(-\gamma,a) = \frac{1}{2\pi i}\int_{-\gamma} \frac{dz}{z-a} = -\frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z-a} = -\eta(\gamma,a).$$

Property 2. $\eta(\gamma$,a) = 0 for all closed curves γ in a disc (or circle) and for all points of a outside the disc.

Proof: $\frac{1}{z-a}$ is analytic inside the disc. (as a lies out side the disc)

Therefore, $\int_{\gamma} \frac{dz}{z-a} = 0$ for all closed curve γ in the disc. (by Theorem 6)

$\eta(\gamma$,a) = $\frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z-a}$ = 0.

Remark: As a point set γ is closed and bounded. Its complement is open. The complement of the point set γ can be represented as a union of disjoint region.

If the complement regions are considered in the extended plane, there is exactly one which contains the point at ∞. Consequently, γ determines one and only one unbounded region.

Property 3:

As a function of 'a' the index $\eta(\gamma$,a) is constant in each of the region determined by γ, and zero in the unbounded region.

Proof:

Join a and b by a line segment not intersecting γ outside the line segment log $\left(\frac{z-a}{z-b}\right)$ is analytic whose derivative is $\frac{1}{z-a} - \frac{1}{z-b}$.

Therefore, $\int_\gamma \left(\frac{1}{z-a} - \frac{1}{z-b}\right)dz = 0 \Rightarrow \int_\gamma \frac{dz}{z-a} = \int_\gamma \frac{dz}{z-b}$

$\Rightarrow \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-b} \Rightarrow \eta(\gamma,\text{a}) = \eta(\gamma,\text{b})$.

If $|a|$ is sufficiently large, γ is contained in a disc $|z| < \rho < |a|$

Therefore by Property (2), $\eta(\gamma,\text{a}) = 0$

$\Rightarrow \eta(\gamma,\text{a}) = 0$ in the unbounded region.

NOTE 1:

We know that, $\int_C \frac{dz}{z-a} = 2\pi i$ where C is a circle about a.

$\Rightarrow$ when γ = C , $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = 1 \Rightarrow \eta(\gamma,\text{a}) = 1$.

Theorem 9: (The Integral Formula)

Suppose that f(z) is analytic in an open set Δ, and let γ be a closed curve in Δ for any point a not on γ. $\eta(\gamma, a)f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-a}$ ----------->(A) where $\eta(\gamma, a)$ is the index a with respect to γ.

Proof: Given f(z) is analytic in open set Δ. Also given that a closed curve γ in Δ and a point a ϵ Δ which does not lie on γ.

Define F(z) = $\frac{f(z)-f(a)}{z-a}$

This function is analytic for z $\neq$ a and for z $= a$, it is not defined.

But it satisfies the condition $\lim_{z\to a}(z-a)F(z) = \lim_{z\to a}[f(z)-f(a)]$
$= f(a) - f(a) \ = 0$

Therefore, By theorem (7) $\int_\gamma \ F(z)dz = 0$---------->(1)

$$\therefore \int_\gamma \left(\frac{f(z)-f(a)}{z-a}\right) dz = 0$$

$$=> \int_\gamma \frac{f(z)dz}{z-a} - \int_\gamma \frac{f(a)dz}{z-a} = 0$$

$$=> \int_\gamma \frac{f(z)dz}{z-a} = f(a)\int_\gamma \frac{dz}{z-a}$$

$$=> \int_\gamma \frac{f(z)dz}{z-a} = f(a)2\pi i \ \eta(\gamma, a)$$

Therefore, $\eta(\gamma, a)f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-a}$

If a $\notin \Delta$, then $\eta(\gamma, a) = 0$

Since $\frac{1}{z-a}$ is analytic in Δ, then we have $\int_\gamma \frac{f(z)dz}{z-a} = 0$ [by Cauchy's theorem for circular disc]

Therefore, $L.H.S = 0 = R.H.S.$ Hence equation (A) is true for all a $\notin \Delta$.

Note: In the special case $\eta(\gamma, a) = 1$, we have $f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-a}$ and this gives a representation formula to compute f(a) as soon as the value of f(z) on γ is given, together with the fact that f(z) is analytic in Δ. This is called Cauchy representation formula. By the change of notation, we write $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)d\xi}{\xi - z}$. This is referred to as Cauchy's integral formula.

Example 1 Compute $\int_{|z|=1} \frac{e^z dz}{z} = 2\pi i$

Solution: $\int_{|z|=1} \frac{e^z dz}{z} = 2\pi i\, \eta(c, 0)\, f(0)$

$= 2\pi i \times 1 \times 1 \quad = 2\pi i$ [since $f(z) = e^z$, $f(0) = e^0 = 1, \eta(c, 0) = 1$]

Example 2 Compute $\int_{|z|=2} \frac{dz}{z^2+1}$ by decomposition of integral into partial fraction.

Solution: $\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$

$$\frac{1}{(z+i)(z-i)} = \frac{A}{z+i} + \frac{B}{z-i}$$

$1 = A(z-i) + B(z+i)$. Put z=i, $B = \frac{1}{2i}$ and z = -i,A=$-\frac{1}{2i}$

$$\int_{|z|=2} \frac{dz}{z^2+1} = \int_{|z|=2} \left[-\frac{1}{2i} . \frac{1}{z+i} + \frac{1}{2i} . \frac{1}{z-i}\right] dz$$

$$= \frac{1}{2i} [-\int_{|z|=2} \frac{dz}{z+i} + \int_{|z|=2} \frac{dz}{z-i}]$$

$= \frac{1}{2i} [-2\pi i \eta(C, -i) f(-i) + 2\pi i \eta(C, i) f(i)]$, where C is the circle, |z| = 2.

$$= \frac{2\pi i}{2i} [-1 + 1] = 0.$$

Example 3. Compute $\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$ under the condition $|a| \neq \rho$.

Solution: Given $|z| = \rho \quad \Rightarrow z = \rho e^{i\theta} \Rightarrow dz = \rho e^{i\theta} i\, d\theta$
$\Rightarrow \overline{dz} = -\rho e^{-i\theta} i\, d\theta$. Then $|dz|^2 = dz\, \overline{dz} = \rho^2 (d\theta)^2$

$$|dz| = \rho\, d\theta \quad = \rho \frac{dz}{ip e^{i\theta}} \quad = \frac{-i\rho dz}{z}$$

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = -i\rho \int_{|z|=\rho} \frac{dz}{z(z-a)(\bar{z}-\bar{a})}$$

$$= -i\rho \int_{|z|=\rho} \frac{dz}{z(z\bar{z} - \bar{a}z - a\bar{z} + a\bar{a})}$$

$$= -i\rho \int_{|z|=\rho} \frac{dz}{z(|z|^2 - \bar{a}z - a\bar{z} + a\bar{a})}$$

$$= -i\rho \int_{|z|=\rho} \frac{dz}{[z\rho^2 - \bar{a}z^2 - z\bar{z}a + a\bar{a}z]} \qquad [as\ |z|^2 = \rho^2]$$

$$= -i\rho \int_{|z|=\rho} \frac{dz}{(z\rho^2 - \bar{a}z^2 - a\rho^2 + a\bar{a}z)}$$

$$= -i\rho \int_{|z|=\rho} \frac{dz}{\rho^2(z-a) - \bar{a}z(z-a)}$$

$$= -i\rho \int_{|z|=\rho} \frac{dz}{(z-a)(\rho^2 - \bar{a}z)}$$

$$= \frac{ip}{\bar{a}} \int_{|z|=\rho} \frac{dz}{(z-a)\left(z - \frac{\rho^2}{\bar{a}}\right)}$$

$$\frac{1}{(z-a)\left(z - \frac{\rho^2}{\bar{a}}\right)} = \frac{A}{(z-a)} + \frac{B}{\left(z - \frac{\rho^2}{\bar{a}}\right)}$$

$$1 = A\left(z - \frac{\rho^2}{\bar{a}}\right) + B(z-a)$$

Put $z = a,\ 1 = A\left(a - \frac{\rho^2}{\bar{a}}\right) = A\frac{1}{\bar{a}}(|a|^2 - \rho^2)$ and $A = \frac{\bar{a}}{|a|^2 - \rho^2}$

Put $z = \frac{\rho^2}{\bar{a}},\ 1 = B\left(\frac{\rho^2}{\bar{a}} - a\right) = B\left(\frac{\rho^2 - |a|^2}{\bar{a}}\right)$ and $B = \frac{\bar{a}}{\rho^2 - |a|^2}$

$$\therefore \int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \frac{i\rho}{\bar{a}} \int_{|z|=\rho} \frac{A}{(z-a)} + \frac{B}{\left(z - \frac{\rho^2}{\bar{a}}\right)}\ dz$$

$$= \frac{i\rho}{\bar{a}}\left[A \int_{|z|=\rho} \frac{dz}{z-a} + B \int_{|z|=\rho} \frac{dz}{z - \frac{\rho^2}{\bar{a}}}\right]$$

$$= \frac{i\rho 2\pi i}{\bar{a}}\left[A\,\eta(C,a)f(a) + B\,\eta\left(C, \frac{\rho^2}{\bar{a}}\right) f\left(\frac{\rho^2}{\bar{a}}\right)\right]$$

$= \frac{-2\pi\rho}{\bar{a}}\left[A\eta(C,a) + B\eta\left(C, \frac{\rho^2}{\bar{a}}\right)\right] \longrightarrow (1)$ where C is $|z| = \rho$

Since , $|a| \neq \rho \quad \Rightarrow |a| < \rho \;\; (or) \;\; |a| > \rho$

Case (i) If $|a| < \rho \Rightarrow$ a lies with in the circle C.

$$\left|\frac{\rho^2}{\bar{a}}\right| = \frac{\rho^2}{|a|} > \frac{\rho^2}{\rho} = \rho. \text{ Therefore, } \frac{\rho^2}{\bar{a}} \text{ lies outside the circle C.}$$

Therefore, $\eta(C, a) = 1,\ \eta\left(C, \frac{\rho^2}{\bar{a}}\right) = 0$

Therefore, (1) becomes $\displaystyle\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \frac{-2\pi\rho}{\bar{a}} A\, \eta(C, a) = \frac{-2\pi\rho}{\bar{a}} A$

$$= \frac{-2\pi\rho}{\bar{a}} \cdot \frac{\bar{a}}{|a|^2 - \rho^2} \qquad [\because |a| \neq \rho]$$

$$= \frac{2\pi\rho}{\rho^2 - |a|^2}$$

Case (ii) If $|a| > \rho \Rightarrow$ a lies outside the circle C.

$$\left|\frac{\rho^2}{\bar{a}}\right| = \frac{\rho^2}{|a|} < \frac{\rho^2}{\rho} = \rho$$

$\Rightarrow \frac{\rho^2}{\bar{a}}$ lies inside the circle C. $\Rightarrow \eta(C, a) = 0, \quad \eta\left(C, \frac{\rho^2}{\bar{a}}\right) = 1.$

Therefore, $\int_{|z|=\rho} \frac{|dz|}{|z-a|} = \frac{-2\pi\rho}{\bar{a}} B\, \eta\left(C, \frac{\rho^2}{\bar{a}}\right)$

$$= \frac{-2\pi\rho}{\bar{a}} \cdot \frac{\bar{a}}{\rho^2 - |a|^2} \qquad = \frac{2\pi\rho}{|a|^2 - \rho^2}$$

HIGHER DERIVATIVES:

The Cauchy integral formula gives us an ideal tool for the study of the local properties of an analytic function.

Lemma 10: Suppose that $\varphi(\xi)$ is continuous on the arc γ. Then the function , $F_n(z) = \int_\gamma \frac{\varphi(\xi)d\xi}{(\xi-z)^n}$ is analytic in each of the region determined by γ, and its derivative is $F_n'(z) = nF_{n+1}(z)$.

Proof: We first prove that F_1(z) is continuous.

Let z_0 be a point not on γ and choose the neighborhood $|z - z_0| < \delta$ so that it does not meet γ. Choose the restricted neighborhood $|z - z_0| < \frac{\delta}{2}$. Then for all $\xi \in \gamma$, $|\xi - z| = |\xi - z_0 + z_0 - z| = |(\xi - z_0) - (z - z_0)|$

$$\geq |\xi - z_0| - |z - z_0| > \delta - {}^{\delta}/_{2} = {}^{\delta}/_{2}$$

Therefore, $|\xi - z| > {}^{\delta}/_{2} \; \forall \; \xi \in \gamma$ ………………………(1)

$$F_1(z) - F_1(z_0) = \int_\gamma \frac{\phi(\xi)d\xi}{(\xi-z)} - \int_\gamma \frac{\phi(\xi)d\xi}{(\xi-z_0)}$$

$$= \int_\gamma \frac{\phi(\xi)[((\xi-z_0)-(\xi-z)]d\xi}{(\xi-z)(\xi-z_0)}$$

$$= \int_\gamma \frac{\phi(\xi)(z-z_0)d\xi}{(\xi-z)(\xi-z_0)}$$

$$|F_1(z) - F_1(z_0)| = \left|\int_\gamma \frac{\phi(\xi)(z-z_0)d\xi}{(\xi-z)(\xi-z_0)}\right|$$

$$\leq \int_\gamma \frac{|\phi(\xi)||z-z_0||d\xi|}{|\xi-z||\xi-z_0|}$$

$$= |z - z_0| \int_\gamma \frac{|\phi(\xi)||d\xi|}{|\xi-z||\xi-z_0|}$$

$$\leq \frac{|z-z_0|^2}{\delta} . \frac{1}{\delta} M \int_\gamma |d\xi| \qquad \text{where } |\phi(\xi)| \leq M$$

$\therefore$ $|F_1(z) - F_1(z_0)| < |z - z_0| \frac{2M}{\delta^2}$L, where L=$\int_\gamma |d\xi|$

As z→ z_0, $\left|F_1(z)- F_1(z_0)\right| \to 0$

$\Longrightarrow F_1(z)$ is continuous at z_0. Since z_0 is arbitrary , $F_1(z)$ is continuous for all z not on γ.

To Prove: $F_1(z)$ is analytic.

$$F_1(z) - F_1(z_0) = (z - z_0) \int_\gamma \frac{\phi(\xi)\, d\xi}{(\xi-z)(\xi-z_0)}$$

$$\Longrightarrow \frac{F_1(z) - F_1(z_0)}{(z-z_0)} = \int_\gamma \frac{\phi(\xi)\, d\xi}{(\xi-z)(\xi-z_0)}$$

$$\Longrightarrow \lim_{z\to z_0} \frac{F_1(z) - F_1(z_0)}{(z-z_0)} = \int_\gamma \lim_{z\to z_0} \frac{\phi(\xi)\, d\xi}{(\xi-z)(\xi-z_0)}$$

$$\lim_{z\to z_0} \frac{F_1(z) - F_1(z_0)}{(z-z_0)} = \int_\gamma \frac{\phi(\xi)\, d\xi}{(\xi-z_0)^2} \quad \Longrightarrow F_1'(z_0) = F_2(z_0)$$

∴ The derivative of $F_1(z)$ exists at z_0 and since z_0 is arbitrary, $F_1(z)$ is analytic for all z.

The general case is proved by induction.

Suppose $F_{n-1}(z)$ is analytic and $F_{n-1}'(z) = (n-1) F_n(z)$

Consider $F_n(z) - F_n(z_0) = \int_\gamma \frac{\phi(\xi)d\xi}{(\xi-z)^n} - \int_\gamma \frac{\phi(\xi)d\xi}{(\xi-z_0)^n}$

$$= \int_\gamma \frac{(\xi-z_0)\phi(\xi)d\xi}{(\xi-z_0)(\xi-z)^n} - \int_\gamma \frac{\phi(\xi)d\xi}{(\xi-z_0)(\xi-z_0)^{n-1}}$$

$$= \int_\gamma \frac{(\varepsilon-z+z-z_0)\phi(\xi)d\xi}{(\xi-z_0)(\xi-z)^n} - \int_\gamma \frac{\phi(\xi)d\xi}{(\xi-z_0)(\xi-z_0)^{n-1}}$$

$$= \int_\gamma \frac{\phi(\xi)d\xi}{(\xi-z_0)(\xi-z)^{n-1}} + \int_\gamma \frac{(z-z_0)\phi(\xi)d\xi}{(\xi-z_0)(\xi-z)^n} - \int_\gamma \frac{\phi(\xi)d\xi}{(\xi-z_0)(\xi-z_0)^{n-1}}$$

$$= \int_\gamma \frac{\phi(\xi)d\xi}{(\xi-z_0)(\xi-z)^{n-1}} - \int_\gamma \frac{\phi(\xi)d\xi}{(\xi-z_0)(\xi-z_0)^{n-1}} + \int_\gamma \frac{(z-z_0)\phi(\xi)d\xi}{(\xi-z_0)(\xi-z)^n} \quad \ldots(2)$$

Take G(ε) = $\frac{\phi(\xi)}{(\xi - z_0)}$, (2) becomes,

$$F_n(z) - F_n(z_0) = \int_\gamma \frac{G(\xi)d\xi}{(\xi - z)^{n-1}} - \int_\gamma \frac{G(\xi)d\xi}{(\xi - z_0)^{n-1}} + (z - z_0)\int_\gamma \frac{G(\xi)d\xi}{(\xi - z)^n} \quad ....(3)$$

Let $G_n(z) = \int_\gamma \frac{G(\xi)d\xi}{(\varepsilon - z)^n}$

Therefore, (3) becomes

$$F_n(z) - F_n(z_0) = G_{n-1}(z) - G_{n-1}(z_0) + \int_\gamma \frac{(z - z_0)G(\xi)d\xi}{(\xi - z)^n}$$

By induction hypothesis applied to G(ξ),

$G_{n-1}(z) \rightarrow G_{n-1}(z_0)$ as z $\rightarrow z_0$ and the factor of z - z_0 is bounded in a neighborhood of z_0. [as $G_{n-1}(z)$ is continuous]

Therefore, $F_n(z) - F_n(z_0) \rightarrow 0$ as z $\rightarrow z_0$.

$\Longrightarrow F_n(z) \rightarrow F_n(z_0) \Longrightarrow F_n(z)$ is continuous at a point z_0.

To Prove: $F_n(z)$ is analytic

$$\frac{F_n(z) - F_n(z_0)}{z - z_0} = \frac{G_{n-1}(z) - G_{n-1}(z_0)}{z - z_0} + \int_\gamma \frac{G(\xi)d\xi}{(\xi - z)^n}$$

$$= \frac{G_{n-1}(z) - G_{n-1}(z_0)}{z - z_0} + \int_\gamma \frac{\phi(\xi)d\xi}{(\xi - z_0)(\xi - z)^n}$$

When z $\rightarrow z_0$, $F_n{'}(z_0) = G_{n-1}{'}(z_0) + \int_\gamma \frac{\phi(\xi)d\xi}{(\xi - z_0)^{n+1}}$

$$F_n{'}(z_0) = G_{n-1}{'}(z_0) + F_{n+1}(z_0)$$

$$= (n-1)\, G_n(z_0) + F_{n+1}(z_0) \ldots.(4) \; [as\;\; G_{n-1}{'}(z_0) = (n-1)\, G_n(z_0)]$$

$$G_n(z_0) = \int_\gamma \frac{G(\xi)d\xi}{(\xi - z_0)^n} = \int_\gamma \frac{\phi(\xi)d\xi}{(\xi - z_0)(\xi - z_0)^n} = \int_\gamma \frac{\phi(\xi)d\xi}{(\xi - z_0)^{n+1}} = F_{n+1}(z)$$

Therefore, (4) becomes, $F_n{}'(z_0) = (n-1)F_{n+1}(z_0) + F_{n+1}(z_0) \; = nF_{n+1}(z_0)$

Since z_0 is arbitrary, $F_n(z)$ is analytic and $F_n{}'(z) = nF_{n+1}(z)$.

Lemma: 11 Prove in detail an analytic function has derivatives of all orders (or) An analytic function defined in a region Ω has derivatives of all orders and these are analytic in Ω.

Proof: Let a ∈ Ω and f(z) be analytic in Ω. Consider a δ-neighbourhood Δ about a and in Δ, for all z inside C, η(C, z) = η(C, a) = 1

Hence by Cauchy's Integral Formula, f(z) = $\frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(\xi)\, d\xi}{(\xi - z)}$

By the above Lemma 10, the integral on the R.H.S is analytic function where

derivative is $\int_C \frac{f(\xi)\, d\xi}{(\xi-z)^2}$. Therefore, $f'(z) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(\xi)\, d\xi}{(\xi-z)^2}$

By the same lemma, the integral on the R.H.S is analytic function. Therefore, whenever f(z) is analytic in Ω then $f'(z)$ is also analytic in Ω.

Therefore, $f''(z) = \frac{2!}{2\pi i} \int_C \frac{f(\xi)\, d\xi}{(\xi-z)^3}$

$\vdots$

$f^n(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)\, d\xi}{(\xi-z)^{n+1}}$ are all analytic functions.

Theorem 12 (Morera's Theorem)

If f(z) is defined and continuous in a region Ω, and $if \int_\gamma f(z)dz = 0$ for all closed curve γ in Ω, then f(z) is analytic in Ω.

Proof: Given $\int_\gamma f(z)dz = 0$ for all closed curve γ in Ω.

=> $\int_\gamma f(z)dz,$ with continuous f, depends only on the end pts of $\gamma.$

=> f is the derivative of the analytic function in Ω.

=> f is analytic in Ω. [by lemma 11]

Theorem 13 (Liouvilles's theorem)

A function which is analytic and bounded in the whole plane must reduce to a constant.

Proof: Let a ∈ Ω and C is any circle of radius r with centre a [$\eta(C, z) = 1$ for all z inside of C]

Now, f(a) = $\frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)}$

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2} \quad \text{[by lemma 3]}$$

Therefore, $|f'(a)| \leq \frac{1}{2\pi}\int_c \frac{|f(z)|\,|dz|}{|z-a|^2}$

$\leq \frac{1}{2\pi}\frac{M}{r^2}\int_c |dz|$ where |z-a| = r and |f(z)| ≤ M for all z∈Ω

.

$$\leq \frac{M}{2\pi r^2}.2\pi r = \frac{M}{r}$$

This is true for all circle with centre a. Letting r → ∞ => $|f'(a)| = 0$

=> $f'(a) = 0$ for all a. => f(z) is a constant.

Theorem 14 (Fundamental theorem of Algebra)

Every polynomial of degree ≥ 1 has atleast one root.

Proof: Let us assume that the polynomial P(z) is of degree n ≥ 1 has no root.

Therefore, P(z) never vanishes in the complex plane.

=> $\frac{1}{P(z)}$ is analytic everywhere => $\frac{1}{P(z)} \to 0 \text{ } as \text{ } z \to \infty$

=> For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\left|\frac{1}{P(z)}\right| < \varepsilon$ for $|z| > \delta$

Since $\frac{1}{P(z)}$ is continuous in the bounded closed domains $|z| \le \delta$.

Therefore, there exists a number K such that $\left|\frac{1}{P(z)}\right| < K \text{ } for \text{ } |z| \le \delta$

Let M = max(ε, K) => $\left|\frac{1}{P(z)}\right| < M$ for all z ∈ Ω

Therefore, by Leouvilles theorem, P(z) is constant.

This is a contradiction.[since P(z) is not a constant]

P(z) must be zero for at least one value of z.

=>The equation P(z) must have at least one root.

Theorem 15 (Cauchy's Estimate)

Let f(z) be analytic in a region Ω and consider a circle C , $|z-a|=\rho$ contained in Ω. If |f(z)| ≤ M on C then $|f^n(a)| \le \frac{n!M}{\rho^n}$

Proof: We know that $f^n(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)dz}{(z-a)^{n+1}}$

$$=>|f^n(a)| \leq \frac{n!}{2\pi}\int_c \frac{|f(z)||dz|}{|(z-a)^{n+1}|}$$

Given |f(z)| ≤ M on C and |z-a| = ρ

$$=>\left|f^{(n)}(a)\right| \leq \frac{n!}{2\pi}\frac{M}{\rho^{n+1}}\int_c |dz| = \frac{n!}{2\pi}.\frac{M}{\rho^{n+1}}2\pi\rho = \frac{Mn!}{\rho^n}$$

This is known as Cauchy’s Estimate.

UNIT V

LOCAL PROPERTIES OF ANALYTIC FUNCTION

5.1 Removable singularities, Taylor's Theorem:

Cauchy's integral formula remains valid in the presece of a finite number of exceptional points, all satisfying the fundamental condition of theorem 5, provided that none of them coincides with a.

Cauchy's formula provides us with a representation of f(z) through an integral which in its dependence on z as the same character at the exceptional points as everywhere else. Points with this character are called removable singularities.

Theorem 1:

Suppose that f(z) is analytic in the region Ω' obtained by omitting point a from a region Ω. A necessary and sufficient condition that there exists an analytic function in Ω which coincides with f(z) in Ω' is that $\lim_{z\to a}(z-a)f(z) = 0$. The extended function is uniquely determined.

Proof: Suppose f(z) is analytic in Ω' and suppose there exists an analytic function F(z) in Ω such that F(z) = f(z) in Ω'.

To prove: $\lim_{z\to a}(z-a)f(z) = 0$

Now, F(z) is analytic in Ω => F(z) is continuous on Ω.

=> F(z) is continuous at a in Ω => given $\varepsilon > 0$ there exists $\delta > 0$ such that |F(z) - F(a) | < ε whenever |z - a| < δ-------->(1)

Now, F(z) = f(z) in Ω'[i.e. z $\neq$ a]

Therefore, (1) becomes |f(z) - A | < ε whenever |z - a| < δ where A = F(a)

$$\lim_{z\to a} f(z) = A \text{ [since F(a) = } \lim_{z\to a} F(z) and\ A = \lim_{z\to a} f(z)]$$

And $\lim_{z\to a}(z-a)f(z) = \lim_{z\to a}(z-a)\lim_{z\to a} f(z)$

$$=0.A=0$$

Let $F_1(z) and\ F_2(z)$ be the extended function in Ω where z ≠ a

$F_1(z) = f(z)$, $F_2(z) = f(z)$ (i.e. in Ω') (i.e. when z ≠ a, $F_1(z) = F_2(z)$)

Moreover, $\lim_{z\to a} f(z) = F_1\,(a)\ and\ \lim_{z\to a} f(z) = F_2\,(a)$

Therefore, $F_1(a) = \lim_{z\to a} f(z) = F_2(a)$. Then $F_1(z) = F_2(z)$ for all z in Ω. Therefore, the extended function is unique.

Conversely, Let a be an exceptional points and $\lim_{z\to a}(z-a)f(z) = 0$

We draw a circle C about a so that C and its inside are contained in Ω.

Therefore, by Cauchy's Integral Formula, f(z) = $\frac{1}{2\pi i}\int_C \frac{f(\xi)d\xi}{\xi - z}$ for all z≠a.

But the integral on the R.H.S represents an analytic function of z throughout the inside of C.

Consequently, the function which is equal to f(z) for z ≠ a and which has the value $\frac{1}{2\pi i}\int_C \frac{f(\xi)d\xi}{\xi - a}$ for z = a, is analytic in Ω and denote it by f(a).

Therefore, the extended function is F(z) = $\begin{cases} f(z)\ for\ z \in \Omega' \\ f(a)\ \text{ for z = a} \end{cases}$

Theorem 2: Taylor's Theorem(Finite development)

If f(z) is analytic in a region Ω, containing a, it is possible to write $f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{n-1}(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)(z-a)^n$, where $f_n(z)$ is analytic in Ω.

Proof: Define $F(z) = \frac{f(z)-f(a)}{z-a}$ for z ≠ a

$\lim_{z\to a}(z-a)F(z) = \lim_{z\to a}(f(z)-f(a))$

$= f(a)-f(a)$ [$\because$ f is analytic in $\Omega \Rightarrow f$ continuous in Ω]
$= 0$

$$\lim_{z\to a} F(z) = \lim_{z\to a}\frac{f(z)-f(a)}{z-a} \quad = f'(a)$$

Hence there exists an analytic function $f_1(z) = \begin{cases} F(z) for\ z \neq a \\ f'(a) for\ z = a \end{cases}$

Repeating this process we can define an analytic function

$$f_2(z) = \begin{cases} \frac{f_1(z)-f_1(a)}{z-a}, z \neq a \\ f_1'(a), \ \ z = a \end{cases}$$ and so on.

The recursive scheme by which $f_n(z)$ is defined and can be written in the form

$$f(z) = f(a) + f_1(z)(z-a)$$

$$f_1(z) = f_1(a) + f_2(z)(z-a)$$

$$\vdots$$

$$f_{n-1}(z) = f_{n-1}(a) + f_n(z)(z-a)$$

From these equations which are trivially valid for z = a and we obtain

$$f(z) = f(a) + (z-a)f_1(a) + (z-a)^2 f_2(a) + \cdots + (z-a)^{n-1} f_{n-1}(a) + (z-a)^n f_n(z)$$

Differentiating n times and setting z = a we get,

$$f^n(a) = n!\, f_n(a) \implies f_n(a) = \frac{f^n(a)}{n!}$$ for all n.

$$\therefore f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)(z-a)^n$$

Note:

This finite development which is the most useful for the study of the local properties of f(z).

Since $f_n(z)$ is analytic in Ω , therefore by Cauchy Integral Formula, we have

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f_n(\xi)d\xi}{(\xi - z)} \text{------------>}(1)$$

where C is the circle about a so that C and its inside are contained in Ω.

Using Taylor's theorem,

$$f_n(\xi) = \frac{f(\xi)}{(\xi - a)^n} - \frac{f(a)}{(\xi - a)^n} - \frac{f'(a)}{1!\,(\xi - a)^{n-1}} - \cdots - \frac{f^{(n-1)}(a)}{(n-1)!}\frac{1}{(\xi - a)}$$

Therefore, (1) becomes,

$$f_n(z) = \frac{1}{2\pi i}\int_C \Big[\frac{f(\xi)}{(\xi-a)^n(\xi-z)} - \frac{f(a)}{(\xi-a)^n(\xi-z)} - \frac{f'(a)}{1!(\xi-a)^{n-1}(\xi-z)} - \cdots\text{. -}$$
$$\ldots\ldots - \frac{f^{(n-1)}(a)}{(n-1)!(\xi-a)(\xi-z)}\Big]d\xi$$

$$= \frac{1}{2\pi i}\int_C \frac{f(\xi)d\xi}{(\xi-a)^n(\xi-z)} - \sum_{r=1}^{n}\frac{f^{(n-r)}(a)}{(n-r)!}\cdot\frac{1}{2\pi i}\int_C \frac{d\xi}{(\xi-a)^r(\xi-z)} \text{------------>}(2)$$

$$Set\ \ F_r(a) = \int_c \frac{d\xi}{(\xi - a)^r(\xi - z)}$$

$$F_1(a) = \int_C \frac{d\varepsilon}{(\xi-a)(\xi-z)} = \frac{1}{z-a}\int_C \frac{(z-a)d\varepsilon}{(\xi-a)(\xi-z)}$$

$$= \frac{1}{z-a}\int_C \frac{[(\xi-a)-(\xi-z)]d\varepsilon}{(\xi-a)(\xi-z)} = \frac{1}{z-a}\int_C \Big(\frac{1}{\xi-z} - \frac{1}{\xi-a}\Big)d\xi$$

$$= \frac{1}{z-a}(2\pi i - 2\pi i) = 0 \text{ identically for all z inside of C.}$$

By lemma 10, $F_n'(a) = nF_{n+1}(a)$

When n = 1, $F_1'(a) = F_2(a)$. Therefore, $F_2(a) = 0$

Similarly, we get $F_r(a) = 0$ for all r ≥ 1

Therefore, from (2) $\quad f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)d\xi}{(\xi-a)^n(\xi-z)}$

This representation is valid inside of C.

Zeroes and poles

Theorem 3:

Let f(z) be defined and analytic in the region Ω. Suppose for some point a ∈ Ω, f(a) and the derivative $f^r(a)$ all vanish. Then f(z) ≡ 0 on Ω.

Proof:

Let C be a circle with centre a and radius R in the region Ω.

By Taylor's theorem,

$$f(z) = f(a) + \frac{f^1(a)}{1!}(z-a) + \frac{f^2(a)}{2!}(z-a)^2 + \cdots + f_n(z)(z-a)^n \text{ for all n}$$

where $f_n(z)$ is analytic in Ω.

By hypothesis, we have f$(z) = f_n(z)(z-a)^n \longrightarrow (1)$ and

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)d\xi}{(\xi-a)^n(\xi-z)}$$

(i.e.) the circumference C has to be contained in Ω in which f(z) is defined and analytic.

$f(\xi)$ is continuous on C and C is compact. $\Rightarrow$ f is bounded on C

(i.e.) $|f(\xi)| \leq M$ on C.

Therefore, $|f_n(z)| \leq \frac{1}{2\pi}\int_c \frac{|f(\xi)||d\xi|}{|(\xi-a)^n||\xi-z|}$

$\leq \frac{M}{2\pi R^n}\int_c \frac{|d\xi|}{|\xi-z|}$ $[as, |\xi - a| = R]$--------------->(2)

$|\xi - z| = |\xi - a + a - z| = |(\xi - a) - (z - a)|$

$\geq |\xi - a| - |z - a| = R - |z - a|$

$\frac{1}{|\varepsilon - z|} \leq \frac{1}{R-|z-a|}$

Therefore, (2) becomes,

$$|f_n(z)| \leq \frac{M}{2\pi R^n(R - |z - a|)}\int_C |d\xi|$$

$$= \frac{M}{2\pi R^n(R-|z-a|)}.2\pi R$$

$$= \frac{M}{R^{n-1}(R-|z-a|)}$$

Therefore, (1) $\implies |f(z)| = |f_n(z)(z-a)^n|$

$$|f(z)| = |f_n(z)||z - a|^n$$

$$\leq \frac{M}{R^{n-1}(R-|z-a|)}.|z-a|^n$$

$$= \left(\frac{|z-a|}{R}\right)^n \frac{MR}{R-|z-a|}$$

Therefore, $0 < \frac{|z-a|}{R} < 1 \quad \Rightarrow (\frac{|z-a|}{R})^n \to 0 \ \ as\ n \to \infty$

$\Rightarrow f(z) = 0$ inside of C ---------->(3)

T.P $f(z) \equiv 0 \; on \; \Omega$

Let E_1 be the set on which f(z) and all its derivatives vanish and E_2 be the set on which the function (or) one of the derivatives different from zero.

E_1 is open, E_2 is open because the functions and all its derivatives are continuous.

Now, $\Omega = E_1 \cup E_2$. But Ω is connected $\Longrightarrow$ either $E_1 = \emptyset$ or $E_2 = \emptyset$

Suppose $E_1 = \emptyset$, then the function and all its derivatives can never vanish at any point. This is a contradiction. $\therefore E_2 = \emptyset \;\Longrightarrow \Omega = E_1 \;\; \therefore$f(z)$\equiv 0$ on Ω.

DEFINITION:

Let f(z) $\not\equiv$ 0. If f(a) = 0 then there exists a least positive integer h such that $f^{(h)}$(a) $\neq$ 0. Then a is the zero of order h.

NOTE: By the previous result, there are no zeros of infinite order.

Lemma 4 If a is a zero of order h then f(z)= $(z-a)^h f_h$(z) where f_h(z) is analytic and f_h(a)$\neq$0.

Proof: Given: f(z) is analytic in Ω. By Taylor's theorem,

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \ldots\ldots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)(z-a)^n$$

$$= (z-a)^h\left[\frac{f^{(h)}(a)}{h!} + \ldots\ldots\ldots\ldots + (z-a)^{n-h} f_n(z)\right]. \qquad [\because f(a)=f'(a)=\ldots\ldots= f^{(h-1)}(a) = 0]$$

f(z) = $(z-a)^h f_h$(z) where f_h(z) = $\frac{f^{(h)}(a)}{h!}$ +……..+ $(z-a)^{n-h} f_n$(z) is analytic.

T.P: f_h(a) $\neq$ 0

f_h(a) = $\frac{f^{(h)}(a)}{h!} \neq 0$.

ISOLATED POINTS:

THEOREM 5:

The zeros of an analytic function which does not vanish identically are isolated (or) The zeros are isolated.

Proof: Let f(z) be analytic function and let f(z) $\not\equiv$ 0. Let z = a be a zero of order h. $\Longrightarrow$ f(z) = $(z-a)^h f_h$(z) where $f_h(z)$ is analytic and $f_h(a) \neq 0$.

Since $f_h(z)$ is continuous, $f_h(z) \neq 0$ in a neighborhood of a and z = a is the only zero of f(z) in this neighborhood.

COROLLARY 6:

If f(z) vanish on a set with an accumulation point in Ω then f(z) $\equiv$ 0

Proof: Let S = {z∈Ω/f(z) = 0} and S has an accumulation point a in Ω.

$\Longrightarrow$ There exists a sequence (a_n) in S such that $a_n \to$ a as n$\to \infty$

$\Longrightarrow$ f(a_n)$\to f(a)$ as n $\to \infty$. Since $(a_n) \in S \Longrightarrow$ f(a_n) = 0 ∀ n $\Longrightarrow$ f (a) = 0.

Claim: f(z)$\equiv$0

Suppose f(z) $\not\equiv$ 0. But then zeros are isolated (ie) not isolated $\Longrightarrow$ not zeros and accumulation point $\Longrightarrow$ not isolated

a is an accumulation point $\Longrightarrow$ a is not a zero of f(z)

$\Longrightarrow$ f(a) $\neq$ 0.This is a contradiction to f(a) = 0.Therefore f(z) $\equiv$ 0 on Ω.

THEOREM 7 (UNIQUENESS THEOREM)

If f(z) and g(z) are analytic in Ω and if f(z) = g(z) on a set which has an accumulation point in Ω, then f(z) $\equiv$ g(z) on Ω.

Proof: Given that f(z) = g(z) on a set which has an accumulation point in Ω

$\Longrightarrow$ f(z) - g(z) = 0 (ie) (f-g)(z) = 0 on a set which has an accumulation point in Ω

By previous corollary, (f-g)(z) $\equiv$ 0 on Ω $\Longrightarrow$ f(z) $\equiv$ g(z) on Ω.

NOTE:

1. If f(z)$\equiv$0 in a sub-region of Ω, then f(z) $\equiv$ 0 on Ω
2. If f(z)=0 on arc $\Longrightarrow$ f(z)$\equiv$0 on Ω
3. An analytic function is uniquely determined by its values on any set with its accumulation point in the region of analyticity [refer uniqueness theorem]

DEFINITION (Isolated Singularity)

Consider a function f(z) which is analytic in neighborhood of a except perhaps at a itself then the point a is called an isolated singularity.

In other words, f(z) shall be analytic in a region o < |z-a| < δ then the point a is called an isolated singularity of f(z).

DEFINITION: (Removable singularity)

If the function is not analytic at ‘a’ but can be made analytic by merely assigning a suitable value to the function at a point a in region Ω.

DEFINITION: (Regular)

If a is the removable singularity and if f(z) is analytic in some(deleted) neighborhood a then f(z) is said to be regular at a.

NOTE: 1. Regular is sometimes used as a synonym for analytic.

1. If a is taken as a Removable singular point then we can define f(a), so that f(z) becomes analytic in the disc |z-a| < δ.

DEFINITION: (Pole)

If f(z) has an isolated singularity at z=a and f(z)→ ∞ at z→a. Then f(z) is said to have a pole at z=a

NOTE:

1. If a is a pole of f(z), we said f(a)= ∞ there exists $\delta' \leq \delta$ such that f(z) ≠ 0 for 0 < |z-a| < δ' [Since f(z) is analytic in region 0 < |z-a| < δ]. In this region the function g(z) = $\frac{1}{f(z)}$ is defined and analytic.

   But the singularity of g(z) at a is removable and g(z) has an analytic extension with g(a) = 0.

   Since g(z) does not vanish identically zero and so a is a zero of g(z) of finite order. We write g(z) = $(z-a)^h g_h$ (z) with g_h(a)≠0.

   The number 'h' is called the order of the Pole and

   f(z) = $\frac{1}{g(z)}$ = $\frac{1}{(z-a)^h g_h(z)}$ = $(z-a)^{-h} f_h(z)$ where $f_h(z)$ = $\frac{1}{g_h(z)}$ is analytic and different from zero in a neighborhood of a.

   DEFINITION: If f(z) has a pole at z = a and f(z) = $(z-a)^h f_h$(z) where f_h (z) is analytic and different from zero in a neighborhood of a. Then h is the order of the pole of f(z) at z = a.

DEFINITION:

A function f(z) which is analytic in a region Ω, except for poles, is said to be meromorphic in Ω.

NOTE:

1. More precisely, to every a ∈ Ω there exists a neighborhood |z-a|<δ contained in Ω, such that f(z) is analytic for 0 < |z-a |< δ and the isolated singularity is a pole.
2. By definition, the poles of a meromorphic function are isolated.
3. The quotient $\frac{f(z)}{g(z)}$ of two analytic function in Ω is a meromorphic function in Ω, provided that g(z) is not identically zero. The only possible poles are the zero of g(z). But a common zero of f(z) and g(z) can also be a removable singularity. In this case the values of the quotient is determined by continuity. The sum, the product, the quotient of the two meormorphic functions are meromorphic.

Detailed discussions of Isolated Singularity:

Consider the condition, (i) $\lim_{z\to a} |z-a|^{\alpha}$|f(z)| = 0

(ii) $\lim_{z\to a} |z-a|^{\alpha}$|f(z)| = ∞ for real values of 'α'.

If (i) holds for certain α, it holds for all larger α and hence for some integer m. Then $(z-a)^m$f(z) has a removable singularity and vanish for z = a.

Either f(z) ≡ 0, in which case (i) holds for all α or $(z-a)^m$f(z) has a zero of finite order k. In the later case it follows at once that

(i) holds for all α > h = m - k
(ii) holds for all α < h.

The discussion shows that there are 3 possibilities:

1) Condition (i) holds for all α and f(z) vanishes identically.

2) There exists an integer h such that (i) holds for all α > h and (ii) for $\alpha <$ h
3) Neither (i) nor (ii) holds for any α.

CASE 1: Trivial

CASE 2: h may be called the algebraic order of f(z) at a. It is positive in the case of a pole and negative in the case of a zero and zero if f(z) is analytic but f(z) not equal to zero at a. The order is always an integer. In the case of a pole of order h, apply Taylor's theorem to an analytic for $(z-a)^h$f(z). We have,

$(z-a)^h$f(z) = B_h + B_{h-1}(z-a) + $B_{h-2}(z-a)^2$ +…………+ $B_1(z-a)^{h-1}$ + φ(z) $(z-a)^h$, where φ(z) is analytic at z = a.

For z ≠ a, we have

f(z) = $B_h(z-a)^{-h}$+$B_{h-1}(z-a)^{-h+1}$+…………+$B_1(z-a)^{-1}$ + φ(z)

The part of this development which proceeds φ(z) is called the singular part of f(z) at z = a.

Therefore, A pole has not only an order but also a well defined singular part.

In the case (3) the point a is an essential singularity. In the neighborhood of an essential singularity f(z) is at the same time unbounded and comes arbitrary close to zero.

Note: The difference of two functions with the same singular part is analytic at a.

CHARACTERIZATION OF THE BEHAVIOUR OF A FUNCTION IN THE NEIGHBORHOOD OF AN ESSENTIAL SINGULARITY:

Theorem 8 (WEIERSTRASS THEOREM)

An analytic function comes arbitrarily closed to any complex value in every neighborhood of an essential singularity.

Proof: If the assertion were not true, we could find a complex number A and a $\delta > 0$ such that |f(z)-A| > δ in a neighborhood of a (except for z = a)

For any $\alpha < 0$, we have then $\lim_{z\to a}|z-a|^{\alpha}|f(z)- A| = \infty$. Hence a would not be an essential singularity of f(z) – A.

Accordingly, there exists a β with $\lim_{z\to a}|z-a|^{\beta}|f(z)- A| = 0$ and we are free to choose $\beta > 0$.

Since $\lim_{z\to a}|z-a|^{\beta}|A| = 0 \Longrightarrow \lim_{z\to a}|f(z)||z-a|^{\beta} = 0$ [$\because$ $||f(z)|-|A|| \geq |f(z)-A|$]

$\Longrightarrow$ a would not be an essential singularity of f(z).

This is a contradiction [as a is an essential singularity of f(z)].

Theorem 9 (LOCAL MAPPING)

Let Z_J be the zeros of a function f(z) which is analytic in a disc Δ and does not vanish identically, each zero being counted as many as its order indicates. For every closed curve γ in Δ which does not pass through a zero $\sum_j \eta(r, z_j) = \frac{1}{2\Pi i}\int_{\gamma} \frac{f'(z)}{f(z)}dz$ where the sum has only a finite number terms not equal to zero.

Proof: Given that f(z) is analytic and not identically zero in an open disc Δ and also given that γ is a closed curve in Δ such that $f(z) \neq 0$ on γ.

Case 1:

If f(z) has only a finite number of zeros in Δ,

Let them be $z_0, z_1, z_2, \ldots\ldots$ where each zero is repeated as many times as its order indicates.

By the repeated application of Taylor's theorem we have, f(z) = (z-z_1) (z-z_2)……………… (z-z_n)g(z) where g(z) is analytic and g(z) ≠ 0 in Δ

Taking log on both sides,

log f(z) = log(z-z_1) + log(z-z_2) +………..+ log(z-z_n) + log g(z).

Differentiate with respect to z,

$$\frac{f'(z)}{f(z)} = \frac{1}{z-z_1} + \frac{1}{z-z_2} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{z-z_n} + \frac{g'(z)}{g(z)}$$

$$\frac{1}{2\Pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \frac{1}{2\Pi i}\int_\gamma \frac{dz}{z-z_1} + \frac{1}{2\Pi i}\int_\gamma \frac{dz}{z-z_2} + \ldots\ldots + \frac{1}{2\Pi i}\int_\gamma \frac{dz}{z-z_n} + \frac{1}{2\Pi i}\int_\gamma \frac{g'(z)}{g(z)}dz$$

$$= \eta(\gamma, z_1) + \eta(\gamma, z_2) + \ldots\ldots\ldots + \eta(\gamma, z_n) + \frac{1}{2\Pi i}\int_\gamma \frac{g'(z)}{g(z)}dz$$

$$= \sum_{i=1}^{n} \eta(\gamma, z_i) + \frac{1}{2\Pi i}\int_\gamma \frac{g'(z)}{g(z)}dz \rightarrow (1)$$

Since g(z) ≠ 0 and g(z) is analytic ⟹ g'(z) is analytic.

Now, $\frac{g'(z)}{g(z)}$ is analytic in Δ and γ is a closed curve in Δ.

By Cauchy's theorem on a disc, $\int_\gamma \frac{g'(z)}{g(z)}dz=0$

∴ (1) becomes $\frac{1}{2\Pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{j=1}^{n} \eta(\gamma, z_j)$

CASE 2: Suppose f(z) has infinitely many zeros in Δ.

It is clear that γ is contained in a concentric disc Δ' smaller than Δ.

If f(z) $\not\equiv 0$, it has only a finite number of zeros in Δ'.

For, If there were infinitely many zeros they would have an accumulation point in the closure of Δ' [By Balzano – Weierstrass theorem]. This is impossible. The zeros outside of Δ' satisfies $\eta(\gamma, z_j) = 0$ and hence do not contribute to the sum in (1). Hence the theorem.

Observation 1: $\sum_j \eta(r, z_j) = \frac{1}{2\pi i}\int_\gamma \frac{df(z)}{f(z)}$ yields a formula for which the total number of zero enclosed by γ.

For, Applying the transformation w = f(z). Let Γ be image of γ under this transformation.

$$\frac{1}{2\pi i}\int_\Gamma \frac{dw}{w} = \frac{1}{2\pi i}\int_\gamma \frac{df(z)}{f(z)}$$

$$= \frac{1}{2\Pi i}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{dz} = \sum_j \eta(r, z_j).\ \text{That is, } \eta(\Gamma,0) = \sum_j \eta(r, z_j).$$

If each $\eta(\gamma, z_j)$ must be either be 0 or 1 then the formula in Theorem 9 $\sum_j \eta(r, z_j) = \frac{1}{2\pi i}\int_\gamma \frac{df(z)}{f(z)}$ yields a formula for which the total number of zero enclosed by γ. This is evidently the case γ is the circle.

Observation 2: Let a be an arbitrary complex value. Apply Theorem 9 to

f(z) - a. The zeros of f(z) - a are the roots of the equation f(z) = a and we denote them by z_j(a).

Therefore, by Theorem 9, $\sum_j \eta(r, z_j(a)) = \frac{1}{2\Pi i}\int_\gamma \frac{f'(z)}{f(z)-a}\mathrm{dz}$

But $\frac{1}{2\pi i}\int_\gamma \frac{d(f(z)-a)}{f(z)-a} = \frac{1}{2\pi i}\int_\Gamma \frac{d(w-a)}{w-a} = \frac{1}{2\pi i}\int_\Gamma \frac{dw}{w-a} = \eta(\Gamma,\mathrm{a})$

$\eta(\Gamma,\mathrm{a}) = \sum_j \eta(r, z_j(a))$. It is necessary to assume that f(z) $\neq$ a on γ.

Observation 3: If a and b are in the same region determined by Γ then $\eta(\Gamma,a) = \eta(\Gamma,b)$ $\Rightarrow \sum_j \eta(r, z_j(a)) = \sum_j \eta(r, z_j(b))$. If γ is a circle, It follows that f(z) takes the values a and b equally many times inside of γ.

THEOREM 10

Suppose that f(z) is analytic at z_0, $f(z_0) = w_0$ and that $f(z) - w_0$ has a zero of order n at z_0. If $\in > 0$ is sufficiently small, there exists a corresponding $\delta > 0$ such that for all a with $|a - w_0| < \delta$ the equation f(z) = a has exactly n roots in the disc $|z - z_0| < \epsilon$.

Proof: We can choose $\in$ so that f(z) is defined and analytic for $|z-z_0| \leq \in$ and so that z_0 is the only zero of $f(z) - w_0$ in this disc. Let γ be a circle $|z-z_0| = \in$ and Γ its image under the mapping w = f(z). Since $w_0 \in$ Complement of the closed set Γ, there exists a neighborhood $|w - w_0| < \delta$ which does not intersect Γ. It follows immediately that all values a in this neighborhood are taken in the same number of times inside γ. The equation $f(z) = w_0$ has exactly n coinciding roots inside of γ, and hence every value a is taken n times. It is understood that multiple roots are counted according to their multiplicity. But if $\in$ is sufficient small, we can assert that all roots of the equation f(z) = a are simple for $a \neq w_0$. Indeed, it is sufficient to choose $\in$ so that $f'(z)$ does not vanish for $0 < |z-z_0| < \in$.

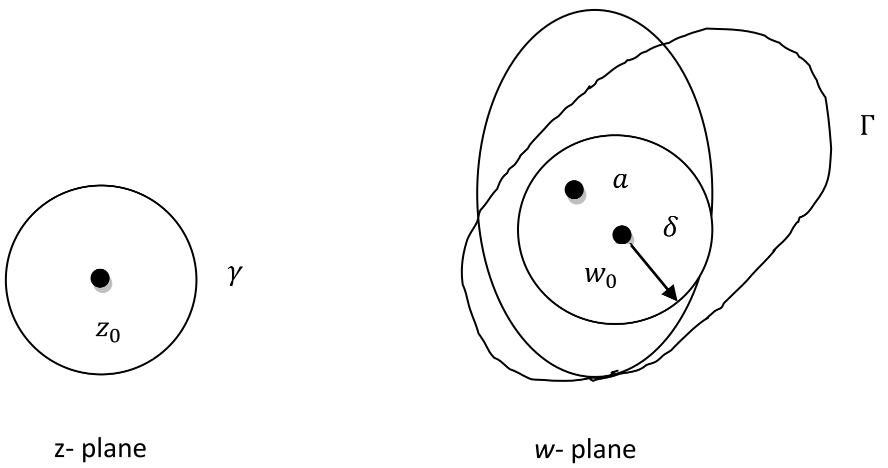

FIG. 5.1 Local correspondence.

Corollary 11

A non constant analytic function maps open sets onto open sets.

Proof: Since the image of every sufficiently small disc $|z-z_0| < \in$ contains a neighborhood $|w - w_o| < \delta$ (By Theorem 10 above)

Therefore, f maps open sets onto open sets. Therefore, f is open map.

Corollary 12

If f(z) is analytic at z_o with $f'(z_0) \neq 0$. It maps a neighborhood of z_0 conformally and topologically onto a region.

Proof: Given $f'(z_0) \neq 0$. Hence n=1, in this case there is a 1-1 correspondence between the disc $|w - w_0| < \delta$ and an open subset Δ of $|z - z_0| < \varepsilon$. Since the open sets in the z-plane corresponds to open sets in the w - plane. The inverse function of f(z) is continuous . Then the mapping is topological. But the mapping can be restricted to neighborhood of z_0 contained in Δ.

$\therefore f'(z_0) \neq 0 \Rightarrow$ f is conformal.

5.2 Maximum Principle

Theorem 13 (The maximum principle)

If f(z) is analytic and non constant in a region Ω, then its absolutely value $|f(z)|$ has no maximum in Ω.

Proof: If $w_0 = f(z_0)$ is any value taken in Ω then by Corollary 11, there exists a neighborhood $|w-w_0| < \delta$ contained in the image of Ω. In this neighborhood there are points of modulus $> |w_0|$.

Hence $|f(z_0)|$ is not the maximum of $|f(z)|$.

Alternative Proof for Maximum Principle:

Let z_0 be any pt in Ω. Consider a circle γ with centre z_0 at radius r.

=> $\xi = z_0 + re^{i\theta}$, $0 \le \theta \le 2\pi$ => $d\xi = ire^{i\theta}\, d\theta$ on γ

By Cauchy Integral formula, $f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)d\xi}{\xi - z_0} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{i\theta})ire^{i\theta}d\theta}{re^{i\theta}}$

$$= \frac{1}{2\pi i}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta \text{ ....(1)}$$

This formula shows that the value of an analytic function at the centre of a circle is equal to arithmetic mean of its values on the circle subject to the condition that the closed disc $|z-z_0| \le r$ is contained in this region of analyticity.

$$\text{(1)} \Rightarrow |f(z_0)| \le \frac{1}{2}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta \text{ ----------(2)}$$

Suppose that $|f(z_0)|$ were a maximum => $|f(z_0+re^{i\theta})| \le |f(z_0)|$

If the strict inequality hold for a single value of θ it would hold, by continuity on a whole arc. Then the mean value of $|f(z_0) + re^{i\theta})|$ would be strictly less than $|f(z_0)|$. Therefore, (2) => $|f(z_0)| < |f(z_0)|$

This is a contradiction. Therefore, f(z) must be constantly equal to $|f(z_0)|$ on all sufficiently small circles $|z-z_0| = r$ and hence in a neighborhood of z_0. => f(z) must reduce to a constant.

Theorem 13'

If f(z) is defined and continuous on a closed and bounded set E and analytic on the interior of E, then the maximum $|f(z)|$ on E is assumed on the boundary of E.

Proof: Since E is compact, $|f(z)|$ has a maximum on E .

Assume $|f(z)|$ has a maximum at z_0.

Case (i): If z_0 is on the boundary, there is nothing to prove.

Case (ii): **A**ssume z_0 is an interior point of E.

Then $|f(z_0)|$ is also a maximum of $|f(z)|$ in a disc $|z-z_0| < \delta$ contained in E.

This is not possible, unless f(z) is constant in the component of the interior of E which contains z_0.

=> By the continuity that $|f(z)|$ = the maximum on the whole boundary of that component.

This boundary is non-empty and it is contained in the boundary of E.

Therefore, the maximum is always assumed at boundary points.

Schwarz's Lemma:

Theorem 14 : If f(z) is analytic for $|z| < 1$ and satisfies the conditions $|f(z)| \leq 1$, $f(0) = 0$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$, then $f(z) = cz$ and with a constant c of absolute value 1.

Proof: Since f(z) is analytic in the disc $|z| < 1$, Taylor's expansion about the origin gives $f(z) = c_0 + c_1 z + c_2 z^2 + \ldots\ldots + c_n z^n + \ldots\ldots$

By hypothesis, $f(0) = 0$ so that $c_0 = 0$

$\therefore f(z) = c_1 z + c_2 z^2 + \ldots\ldots + c_n z^n + \ldots\ldots$ -----------(1)

Consider the function, $f_1(z) = \frac{f(z)}{z} = c_1 + c_2 z + \ldots\ldots$ in the unit disc $|z| < 1$, $f_1(z)$ is analytic. If we set $f_1(0) = c_1 = f'(0)$

(i.e) $f_1(z) = \begin{cases} \frac{f(z)}{z} \text{ if } z \neq 0 \\ f'(0) \text{ if } z = 0 \end{cases}$

Let z = a be an arbitrary point of the unit disc.

We choose 'r' such that $|a| < r < 1$

Since $|f(z)| \leq 1$, on the circle $|z| = r$, we have $|f_1(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r}$ ---------(2)

By the maximum principle , the inequality (2) also holds in this disc $|z| \leq r$

$\therefore\ |f_1(a)| = \left|\frac{f(a)}{a}\right| \leq \frac{1}{r}$ If we let $r \to 1$ we see that $|f_1(a)| \leq 1$

(i.e) $\left|\frac{f(a)}{a}\right| \leq 1$ (or) $|f(a)| \leq |a|$

Since a is arbitrary ,we have $|f(z)| \leq |z|\ \forall z$ ……….(3) for which $|z| < 1$

[In particular, $|f_1(0)| = |f'(0)| \leq 1$ (given)]

If the equality in (3) holds at a single point it means that $f_1(z)$ attains its maximum and hence that $f_1(z)$ must reduce to a constant [by maximum modulus principle]

Therefore, $|f_1(a)| = 1$ can hold only when $f_1(z) = \frac{f(z)}{z} = e^{i\alpha}$ (i.e) $f(z) = ze^{i\alpha}$ where α is a real constant or $f(z) = cz$ where $|c| = |e^{i\alpha}| = 1$. Hence the theorem.

Cycles and chains

Definition : Chains

Consider the formal sums $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \ldots + \gamma_n$ which need not be an arc and we can define the corresponding integral $\int_{\gamma_1+\gamma_2 \ldots +\gamma_n} f dz = \int_{\gamma_1} f dz + \int_{\gamma_2} f dz + \cdots \int_{\gamma_n} f dz$ such formal sums $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \ldots + \gamma_n$ of arcs are called chains.

Definition : Cycles

A chain is a cycle if it can be represented as a sum of closed arcs (or) a chain is cycle if and only if in any representation the initial and end pts of the individual arcs are identical in pairs.

Definition :

A region is simply connected if its complement with the extended plane is connected

Definition: (Homology)

A cycle γ in an open set Ω is said to be homologues to zero with respect to Ω if $\eta(\gamma,a) = 0$ for all points a in the complement of Ω.

The General statement of Cauchy's Theorem

If f(z) is analytic in Ω then $\int_{\gamma} f(z)dz = 0$ for every cycle γ which is homologues to zero.

5.3 The calculus of Residues

Now the determination of line integrals of analytic functions over a closed curve can be reduced to the determination of a period. We are thus possessing of a method which in many cases permits to evaluate integrals without resorting to explicit calculus.

In order to make this method more systematic a simple formulation known as the calculus of residues was introduced by Cauchy.

Residue Theorem:

All the results which were derived as consequences of Cauchy's theorem for a disc remains valid in arbitrary region for all cycles which are homologues to zero.

Cauchy's Integral formula can be expressed in the form, if f(z) is analytic in a region Ω.

$\eta(\gamma,a)\, f(a) = \frac{1}{2\Pi i}\int_{\gamma} \frac{f(z)}{z-a}$ dz for every cycle γ which is homologus to zero in Ω .

We now turn to the discussion of a function f(z) which is analytic in a region Ω except for isolated singularities.

Let us assume that there are only finite number singular points denoted by

a_1 ,a_2 ,…….a_n. The region obtained by excluding the points a_j will be denoted by Ω' . To each a_j, there exists a $\delta_j > 0$ such that the doubly connected region is contained in Ω'. Draw a circle C_j about a_j of radius less than δ_j .

Let $P_j = \int_{c_j} f(z)dz$ be the corresponding period of f(z).

The particular function $\frac{1}{z-a_j}$ has the period $2\pi i$.

Therefore, Set $R_j = \frac{P_j}{2\Pi i}$ ---------(1)

Now, f(z) - $\frac{R_j}{z-a_j}$ has a vanishing period. The constant R_j which produces this result is called the residue at the point a_j.

Definition: The residue of f(z) at an isolated singularity a is the unique complex number R which makes f(z) - $\frac{R}{z-a}$ the derivative of a single valued analytic function in an annulus $0 < |z-a| < \delta$.

Note: Notation $R = \underset{z=a}{Res} f(z)$. Since $R_j = \frac{1}{2\Pi i} \int_{c_j} f(z)dz$ where C_j is the circle about the isolated singular point a_j , $\underset{z=a_j}{Res} f(z) = \frac{1}{2\Pi i} \int_{c_j} f(z)dz$

Theorem 15: (Residue Theorem):

Let f(z) be analytic except for isolated singularities a_j in a region Ω then

$\frac{1}{2\Pi i} \int_{\gamma} f(z)dz = \sum_j \eta(\gamma, a_j) \underset{z=a_j}{Res} f(z)$ for any cycle γ which is homologus to zero in Ω and does not pass through any points a_j.

Proof:

Case(i): Suppose there exists only finite number of isolated singularities (say) $a_1, a_2, \ldots\ldots, a_n$.

Define $\Omega' = \Omega - \bigcup_{i=1}^{n}\{a_i\}$. Therefore, f(z) is analytic in the region Ω'.

Let γ be a cycle homologues to zero in Ω and does not pass through any one of the a_j's.

Let C_j be a circle with centre a_j and radius > 0. Then $\frac{1}{2\Pi i}\int_{c_j} f(z)dz$ is defined as the residue of f(z) at a singularity a_j . Consider a cycle $\Gamma = \gamma - \sum_j \eta(\gamma, a_j)C_j$,

Now, $\eta(\mathbf{\Gamma},a_k) = \eta(\gamma-\sum_j \eta(\gamma, a_j)C_j, a_k) = \eta(\gamma, a_k) - \eta(\sum_j \eta(\gamma, a_j)C_j ,a_k)$

$$= \eta(\gamma, a_k)- \sum_j \eta(\gamma, a_j) \eta(C_j,a_k) = \eta(\gamma,a_k) - \eta(\gamma,a_k) \eta(C_k, a_k)$$

$$= \eta(\gamma,a_k) - \eta(\gamma,a_k) = 0$$

Thus, $\eta(\mathbf{\Gamma},a_k) = 0$

Let a does not belongs to Ω. $\eta(\mathbf{\Gamma},a) = \eta(\gamma-\sum_j \eta(\gamma, a_j)C_j, a)$

$$= \eta(\gamma,a) - \eta(\sum_j \eta(\gamma, a_j)C_j ,a)= \eta(\gamma,a) - \sum_j \eta(\gamma, a_j) \eta(C_j ,a) = 0 – 0 = 0$$

Therefore, $\boldsymbol{\Gamma}$ is a cycle homologues to zero which does not pass through the a_j's. (i. e) $\mathbf{\Gamma}$ is a cycle homologues with respect to Ω'.

Therefore, by Cauchy general theorem,

$$\int_{\Gamma} f(z)dz = 0 \Rightarrow \int_{\gamma-\sum_j \eta(\gamma,a_j)\mathrm{Cj}} f(z)dz = 0$$

$$\Rightarrow \int_{\gamma} f(z)dz = \sum_j \eta(\gamma, a_j) \int_{C_j} f(z)dz$$

$$\Rightarrow \frac{1}{2\Pi i}\int_{\gamma} f(z)dz = \sum_j \eta(\gamma, a_j) \frac{1}{2\Pi i} \int_{C_j} f(z)dz$$

$$= \sum_j \eta(\gamma, a_j) \underset{z=a_j}{Res} f(z)$$

Case (ii): Suppose there exists infinitely many isolated singularities to **Ω**. The set of points a such that η(γ, a) = 0 is open . Since γ is compact there is a large circular disc D such that $\gamma \subset \overline{D}$ and for $a \epsilon \sim\overline{D}$, η(γ,a) = 0.

Then the set of points b such that η(γ,b) ≠ 0 equal to $\overline{D}$-A = $\overline{D} \cap A^c$ where

A = {a ϵ $\overline{D}$ / η(γ,a) = 0}.

Therefore, $\overline{D}$ - A is a closed and bounded set and hence $\overline{D}$ - A is compact.

Hence, there exists only a finite number of a_j such that η(γ,a_j) ≠ 0 and for this a_j case(i) applies.

Therefore, $\frac{1}{2\Pi i}\int_{\gamma} f(z)dz = \sum_j \eta(\gamma, a_j) \underset{z=a_j}{Res} f(z)$

Note1: In the applications, it is frequently the case that each η(γ,a_j) is either 0 or 1 . we have $\sum_j \underset{z=a_j}{Res} f(z)$,where the sum is extended over all singularities enclosed by γ.

2) The residue of f(z) at a simple pole z = a is $\underset{z \to a}{lim} (z-a)$f(z) = 0.

Problem 1:

Find the poles and the residues at their poles of the following function $\frac{e^z}{(z-a)(z-b)}$.

Solution: Let f(z) = $\frac{e^z}{(z-a)(z-b)}$

Poles of f(z) is given by (z-a)(z-b)=0 (i.e) z = a or z = b

(i.e.) z = a and z = b are the poles of f(z).

Residue of f(z) at the simple pole z = a

$$= \lim_{z \to a} (z-a)\frac{e^z}{(z-a)(z-b)} = \frac{e^a}{a-b}$$

Residue of f(z) at the simple pole z = b

$$= \lim_{z \to b}(z-b)f(z)$$

$$= \lim_{z \to b}(z-b)\frac{e^z}{(z-a)(z-b)} \quad = \frac{e^b}{b-a}.$$

When b = a, $\quad f(z) = \frac{e^z}{(z-a)^2} \quad => z = a$ is a pole of order 2.

We know that if z = a is a pole of order h and if

f(z) = $z_h(z-a)^{-h} + \cdots + z_1(z-a)^{-1} + \varphi(z)$ then $Resf(z)_{z=a} = z_1$

By Taylor's theorem for $g(z) = e^z$

$$g(z) = g(a) + \frac{g'(a)}{1!}(z-a) + g_2(z)(z-a)^2$$

$e^z = e^a + \frac{e^a}{1!}(z-a) + g_2(z)(z-a)^2$ where $g_2(z)$ is analytic

Divide by $(z-a)^2$

$$\frac{e^z}{(z-a)^2} = \frac{e^a}{(z-a)^2} + \frac{e^a}{(z-a)} + g_2(z)$$

$\int_\gamma \frac{e^z}{(z-a)^2}dz = \int_\gamma \frac{e^a}{(z-a)^2}dz + \int_\gamma \frac{e^a}{(z-a)}dz + \int_\gamma g_2(z)dz$ where C is the circle with centre a.

$$= e^a \int_\gamma \frac{dz}{z-a} \text{ , } where \quad \gamma \sim 0$$

$$= e^a . 2\pi i$$

$$Res_{z=a}\frac{e^z}{(z-a)^2} = \frac{1}{2\pi i}\int_\gamma f(z)dz$$

$$= \frac{1}{2\pi i}e^a . 2\pi i \quad = e^a$$

The Argument principle:

Theorem 16:

If f(z) is meromorphic in Ω with zeros a_j and poles b_k. Then

$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_j \eta(\gamma, a_j) - \sum_k \eta(\gamma, b_k)$ for every cycle γ which is homologues to zeros in Ω which does not pass through any of the zeros or poles.

Proof: Let a_j be the zero of f(z) of order h.

Therefore, $f(z) = (z-a)^h f_h(z)$, where $f_h(a_j) \neq 0$ and $f_h(z)$ is analytic.

Taking log and differentiate with respect to z.

$$\frac{f'(z)}{f(z)} = \frac{h}{(z-a_j)} + \frac{{f_h}'(z)}{f_h(z)}$$

Therefore, $z = a_j$ is a simple pole of $\frac{f'}{f}$ with residue h.

Let b_k be the pole of f(z) of order l.

=> $f(z) = (z-b_k)^{-l}\sigma_{(z)}$ where $\sigma(b_k) \neq 0$ and $\sigma(z)$ is analytic.

Taking log and differentiate with respect to z

$$\frac{f'(z)}{f(z)} = -\frac{l}{z-b_k} + \frac{\sigma'(z)}{\sigma(z)}$$

Therefore, $z = b_k$ is a simple pole of the function $\frac{f'}{f}$ with residue – l .

By Residue theorem, we have $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_j \eta(\gamma, a_j) - \sum_k \eta(\gamma, b_k)$.

Corollary 17 (Rouche's theorem)

Let γ be a cycle homologous to zero in Ω and such that $\eta(\gamma, z)$ is either zero or one for any point z not on γ. Suppose that f(z) and g(z) are analytic in Ω and satisfy the inequality $|f(z) - g(z)| < |f(z)|$ $on\, \gamma$. Then f(z) and g(z) have the same number of zeros enclosed by γ.

Proof: Given that f and g are analytic in Ω.

Further, f and g do not have a zero on γ.

For, f has a zero a on $\gamma \Rightarrow f(a) = 0$ $for\ a \in \gamma$

By hypothesis, $|f(a) - g(a)| < |f(a)| \Rightarrow |g(a)| < 0$.This is a contradiction.

Therefore, f has no zero on γ. Similarly, g has no zero on γ.

$\therefore$ f and g can not have a zero on γ. Let $F(z) = \frac{g(z)}{f(z)}$

$\Rightarrow F(z)$ is meromorphic in Ω.

Let N = number of zeros of F(z) enclosed by γ.

= number of zeros of g(z) inside of γ.

Let P = number of poles of F(z) enclosed by γ.

= number of zeros of f(z) inside of γ.

Therefore, By the Argument principle, $N - P = \frac{1}{2\pi i}\int_\gamma \frac{F'(z)}{F(z)}dz$

$= \eta(\Gamma, 0)$ where $\Gamma = F(\gamma)$

Given $|f(z) - g(z)| < |f(z)|, \quad z \in \gamma$

$=> \left|1 - \frac{g(z)}{f(z)}\right| < 1, \quad z \in \gamma$

=> | 1 - F(z) | < 1 , z ∈ γ

=> $\Gamma = f(\gamma)$ which is contained in the open unit circular disc of centre one and radius 1.

=> $\eta(\Gamma, 0) = 0$ => N - P = 0 => N = P

=> Number of zeros of g (z) inside of γ = Number of zeros of f (z) inside of γ

Therefore, f and g have the same number of zeros enclosed by γ.

REMARK: f(z) and g(z) are interchangeable. Therefore, we have the condition | f(z) – g(z) | < | g(z) | => f and g have the same number of zeros.

Take $\emptyset(z) = f(z) - g(z)$ => $\emptyset(z) + g(z) = f(z)$

Therefore, $|\emptyset(z)| < |g(z)|$

Therefore, g(z) and g(z)+ $\emptyset(z)$ *have the same number of zeros*

Therefore, *if* $|\emptyset(z)| < |g(z)|$

then g(z) and g(z)+ $\emptyset(z)$ *have the same number of zeros*.

Problem 1:

How many roots does the equation $z^7-2z^5+6z^3-z+1=0$ have in the disc $|z|<1$

Solution:

Of the coefficients 1,-2,6,-1,1,the coefficients 6 has the maximum absolute value.

To use Rouche's theorem ,

Let $f(z) = 6z^3$ and $g(z) = z^7 - 2z^5 - z + 1$

On $|z|=1$, $|f(z)| = 6\,z^3=6$

$$|g(z)| = |z^7 - 2z^5 - z + 1|$$

$$< |z|^7+2|z|^5+|z|+1 = 5$$

=> $|g(z)| < |f(z)|$. But $f(z)= 6\,z^3$ has 3 roots $z = 0$

By Rouche's theorem,

=> g + f and f have the same number of zeros inside the circle $|z|=1$

=> $g+f = z^7-2z^5+6z^3-z+1 = 0$ has 3 roots inside of the circle $|z|< 1$

Problem 2. How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2.

Solution: Consider the circle $|z| = 1$. Of the coefficients 1,-6,3

6 has the maximum absolute value.

To use the Rouche's theorem, take $f(z)= -6z$ and $g(z)= z^4+3$

On $|z|=1$, $|f(z)| = 6|z| = 6$ and $|g(z)| = |z^4+3| \leq |z^4|+3 = 4$

Therefore, $|g(z)| < |f(z)|$

But $f(z) = 6z$ has one root $z = 0$ inside of $|z|=1$

By Rouche's theorem

=> g+f and f have the same number of zeros inside the circle $|z| = 1$

=> $g+f = z^4-6z+3=0$ has one root inside the circle $|z| = 1$

Consider the circle $|z| = 2$

Let f(z) = z^4 and g(z) = -6z+3 . On $|z| = 2$,

$|f(z)| = |z|^4 = 16$ and $|g(z)| \leq 6|z|+3 \leq 15$

Therefore, $|g(z)| < |f(z)|$. But f(z) = z^4

Therefore, f(z) has 4 roots z = 0 inside the circle $|z| = 2$

Hence by Rouche's theorem ,

f + g and f have the same number of zeros inside of $|z| = 2$

Therefore, f(z) + g(z) = z^4 - 6z + 3 = 0 has 4 roots inside of $|z| < 2$

Therefore, the number roots lying between $|z| = 1$ and $|z| = 2$ is 4 – 1 = 3

5.4 Evaluation of definite integrals:

Type 1:

$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$ *where the integrand* $R(\cos\theta, \sin\theta)\, d\theta$ is a rational function of $\cos\theta$ *and* $\sin\theta$

Put z = $e^{i\theta}$.$\Longrightarrow$ dz = $ie^{i\theta}\, d\theta \Longrightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

$\cos\theta = \frac{z+1/z}{2}$, $\sin\theta = \frac{z-1/z}{2i}$

Also $|z|$=1 (i.e) C is the unit circle $|z| = 1$

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = \int_C R\left(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\right)\frac{dz}{iz} = \int_C f(z)dz$$
$$=2\pi i\ (sum\ of\ residues\ of\ f(z) at\ the\ poles\ with\ in\ C)$$

Problem 1

Compute $\int_0^{\pi} \frac{d\theta}{a+\cos\theta}$, a > 1

Solution: $\int_0^{\pi} \frac{d\theta}{a+\cos\theta} = ½ \int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$

since $\cos\theta$ *takes the same values in the interval* $(0,\pi)$ *and* $(\pi,2\pi)$

Put z = $e^{i\theta}$ $\Longrightarrow$ dz = i$e^{i\theta}$dθ => dθ = $\frac{dz}{ie^{i\theta}}$ = $\frac{dz}{iz}$

$\int_0^{2\pi} \frac{d\theta}{a+cos\theta} = \int_C \frac{dz/iz}{a+\frac{1}{2}(z+\frac{1}{z})}$ where C is the unit circle |z| < 1

$$= \int_C \frac{dz}{iz(\frac{2az+z^2+1}{2z})}$$

$$= -i2\int_C \frac{dz}{z^2+2az+1}$$

$= -2i\int_C f(z)dz$,where f(z) = $\frac{1}{z^2+2az+1}$

The poles of f(z) are given by $z^2 + 2az + 1 = 0$

$$Z = \frac{-2a \pm \sqrt{(4a^2-4)}}{2}$$

$$= \frac{2(-a \pm \sqrt{a^2}-1)}{2}$$

α = - a + $\sqrt{a^2-1}$ *and* β = - a - $\sqrt{a^2-1}$

αβ = $a^2 - (a^2-1) = 1$. But |α||β|=1 => |α| = $\frac{1}{|\beta|}$

Now, |β | > 1 => $\frac{1}{|\beta|}$ < 1 => |α| < 1

α lies inside the circle C, |z| = 1.

$\text{Res}_{z=\alpha} f(z) == \lim_{z\to\alpha}(z-\alpha)f(z) \quad = \quad \lim_{z\to\alpha} \frac{z-\alpha}{(z-\alpha)(z-\beta)}$

$$= \frac{1}{\alpha-\beta} = \frac{1}{-a+\sqrt{a^2-1}-(-a-\sqrt{(a^2-1)}} = \frac{1}{2\sqrt{(a^2-1)}}$$

$I = \frac{2}{i}\int_C \ f(z)dz$

$=\frac{2}{i}.2\pi i$ [sum of the residue of the poles with in C]

$= 4\pi\ (\underset{z=a}{Res} f(z)) = 4\pi \frac{1}{2\sqrt{(a^2-1)}}$

Therefore, $\int_0^{2\pi} \frac{d\theta}{a+cos\theta} = \frac{2\pi}{\sqrt{(a^2-1)}}$

$\int_0^{\pi} \frac{d\theta}{a+cos\theta} = 1/2 \int_0^{2\pi} \frac{d\theta}{a+cos\theta} = 1/2 \frac{2\pi}{\sqrt{(a^2-1)}} = \frac{\pi}{\sqrt{(a^2-1)}}$

Deduction: $\int_0^{2\pi} \frac{d\theta}{a+cos\theta} = \frac{2\pi}{\sqrt{(a^2-1)}}$

Diff. w.r.to a , $-\int_0^{2\pi} \frac{d\theta}{(a+cos\theta)^2} = -1/2 \frac{2\pi}{(a^2-1)^{\frac{3}{2}}} .2a$

$\int_0^{2\pi} \frac{d\theta}{(a+cos\theta)^2} = \frac{2a\pi}{(a^2-1)^{\frac{3}{2}}}$

Problem 2

Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{a+sin^2x}$, $|a| > 1$

Solution:

Take $I = \int_0^{\frac{\pi}{2}} \frac{dx}{a + sin^2x} = \int_0^{\frac{\pi}{2}} \frac{dx}{a+\frac{1-cos2x}{2}}$

Put $\theta = 2x$. when $x = 0$, $\theta = 0$ and when $x = \frac{\pi}{2}$, $\theta = \pi$ then $d\theta = 2\ dx$

Let $I = \int_0^{\pi} \frac{d\theta}{2a+1-\cos\theta} = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{2a+1-\cos\theta}$

Put $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$ and $d\theta = \frac{dz}{iz}$

Let C be the unit circle |z | = 1

$$I = \frac{1}{2} \int_C \frac{\frac{dz}{iz}}{2a+1-\left(z+\frac{1}{z}\right)/2}$$

$$=1/2\int_C \frac{dz}{iz\left(\frac{4a+2-(z+\frac{1}{z})}{2}\right)}$$

$$=-1/i\int_C \frac{dz}{z^2-2(2a+1)z+1}$$

$$=i\int_C \frac{dz}{z^2-(4a+2)z+1}$$

$=i\int_C f(z)dz$ where $f(z) = \frac{1}{z^2-(4a+2)z+1}$

= i 2πi (sum of the residues of f(z) at the poles with in C).

The poles of f(z) is given by $z^2 - (4a + 2)z + 1 = 0$

$$Z = \frac{(4a+2)\pm\sqrt{((4a+2)^2-4)}}{2} = \frac{(4a+2)\pm\sqrt{(16a^2+4+16a-4}}{2}$$

$$= \frac{4a+2 \pm 4\sqrt{(a^2+a)}}{2} \qquad = 2a+1 \pm 2\sqrt{(a^2+a)}$$

Let $\alpha = 2a+1+2\sqrt{(a^2+a)}$ *and* $\beta = 2a+1-2\sqrt{(a^2+a)}$

$|\alpha| = |2a+1+2\sqrt{(a^2+a)}| > 1.$ *Now,* $\alpha\beta = 1 \implies |\alpha||\beta| = 1$

$\Rightarrow |\beta| = \frac{1}{|\alpha|} < 1$, $|\beta| < 1$. Therefore, β lies with in the circle $C: |z| = 1$

$\underset{z=\beta}{Res} f(z) = \lim_{z\to\beta}(z-\beta)f(z)$

$= \lim_{z\to\beta}(z-\beta)\frac{1}{(z-\alpha)(z-\beta)}$

$= \frac{1}{(\beta-\alpha)} \quad = \frac{1}{(2a+1)-2\sqrt{a^2+a}-\left((2a+1)+2\sqrt{a^2+a}\right)} \quad = \frac{1}{-4\sqrt{a^2+a}}$

$I = i\int_c f(z)dz$

$= i \times 2\pi i$ (sum of the residue at the poles with in C)

$= -2\pi \underset{z=\beta}{Res} f(z) \quad = -2\pi\left(\frac{1}{-4\sqrt{a^2+a}}\right) \quad = \frac{\pi}{2\sqrt{a^2+a}}$

Note:

If $z = a$ is a pole of order m then $\underset{z=a}{Res} f(z) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$

For,

Since a is a pole order m, $f(z) = a_0 + a_1(z-a) + \cdots + \frac{b_1}{z-a} + \cdots\cdots + \frac{b_m}{(z-a)^m}$

$\{(z-a)^m f(z)\} = a_0(z-a)^m + a_1(z-a)^{m+1} + \cdots + b_1(z-a)^{m-1} + \cdots + b_m$

$$\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} = (m-1)(m-2)\cdots\cdots 1.b_1 = b_1(m-1)!$$

Therefore, $b_1 = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$

Therefore, $\underset{z=a}{Res} f(z) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$

Problem 3. Apply the Cauchy's residue theorem and prove that

$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta = \frac{2\pi}{n!}$

Solution: $I = \int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\, d\theta$

$= \text{Real part of } \int_0^{2\pi} e^{\cos\theta}\, e^{i(sin\theta - n\theta)}\, d\theta$

$= \text{Real part of} \int_0^{2\pi} e^{\cos\theta + isin\theta}\, e^{-in\theta}\, d\theta$

$= \text{Real part of} \int_0^{2\pi} e^{e^{i\theta}}\, e^{-in\theta}\, d\theta$

Put $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$ and C is a unit circle $|z| = 1$

$I = \text{Real part of} \int_0^{2\pi} e^{z}\, \frac{1}{z^n}\frac{dz}{iz}$

$= \text{Real part of} \frac{1}{i}\int_C \frac{e^z}{z^{n+1}} dz$

$= \text{Real part of} \frac{1}{i}\int_C f(z)dz$ where $f(z) = \frac{e^z}{z^{n+1}}$

$= \text{Real part of} \frac{1}{i} \times 2\pi i \times Sum\ of\ residues\ at\ the\ poles\ with\ in\ C$

Poles of $f(z)$

$z = 0$ is a pole of order $n + 1$ lies within C.

$\underset{z=0}{Res}\ f(z) = \lim_{Z\to 0} \frac{1}{n!}\frac{d^n}{dz^n}(z^{n+1}\text{f(z)}) = \lim_{Z\to 0} \frac{1}{n!}\frac{d^n}{dz^n}(z^{n+1}.\frac{e^z}{z^{n+1}}) = \frac{1}{n!}.$

I = R.P. $of\ \frac{1}{i}\int_C\ f(z)dz$ = R.P.$of\ \frac{1}{i}.2\pi i$(Sum of residues at the poles within C)

= R.P $2\pi.\frac{1}{n!} = \frac{2\pi}{n!}$

Lemma 18:

If $\lim\limits_{Z\to a}$(z-a)f(z) = A and if C is the arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z - a|$ = r then $\lim_{r\to 0}\int_C\ f(z)dz$ = iA(θ_2-θ_1)

Proof:

$\lim\limits_{z\to a}$(z-a)f(z) = A $\Rightarrow$ given $\varepsilon > 0$,there exists $a\ \delta > 0$ depending on ε such that

$|(z-a)f(z) - A| < \varepsilon$ for $|z-a| < \delta$. But $|z-a|$ = r

Therefore, if r < δ then $|(z-a)f(z) - A| < \varepsilon$ on the arc C.

Therefore, (z-a)f(z) = A + η where $|\eta| < \varepsilon \Rightarrow$ f(z) = $\frac{A+\eta}{z-a}$

Therefore, $\int_C f(z)dz = \int_c \frac{A+\eta}{z-a}dz$ = (A+ η) $\int_C \frac{dz}{z-a}$

= (A+ η) $\int_{\theta_1}^{\theta_2} \frac{re^{i\theta}.id\theta}{re^{i\theta}}$ where |z| = r.

= i(A+ η)(θ_2 - θ_1)

= i A(θ_2 - θ_1) + i η(θ_2 - θ_1)

$\left|\int_C f(z)dz - iA(\theta_2 - \theta_1)\right| = |i\eta(\theta_2 - \theta_1)| = |\eta||\theta_2 - \theta_1| < \varepsilon(\theta_1 - \theta_2)$

Since $\varepsilon \to 0 \Rightarrow r \to 0$. Therefore, $\lim_{r\to a} \int_C f(z)dz$ = iA $(\theta_2 - \theta_1)$

NOTE:

1) In particular, if A = 0 then $\lim_{z\to 0} \int_C f(z)dz = 0$

2) $\lim\limits_{r\to 0}$(z-a)f(z) = $Res_{z=a}$f(z)

Lemma 19:

If C is the arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z|$ = R then $\lim_{R\to\infty} zf(z)$=A then

$\lim\limits_{R \to \infty} \int_C f(z)dz$ = iA$(\theta_2 - \theta_1)$

Proof:

$\lim\limits_{R \to \infty}$ zf(z) = A.Therefore we choose R so large that |zf(z) – A| < ε

On the arc C, (i.e) zf(z) = A + η where $|\eta|$ < ε

Therefore $\int_C f(z)dz$ = $\int_c \frac{A+\eta}{Re^{i\theta}}$.Ri$e^{i\theta} d\theta$

=i(A+ η) $(\theta_2 - \theta_1)$

= Ai$(\theta_2 - \theta_1)$+ ηi $(\theta_2 - \theta_1)$

$$\left|\int_C f(z)dz - iA(\theta_2 - \theta_1)\right| = |i\eta(\theta_2 - \theta_1)|$$

$$= |\eta|(\theta_2 - \theta_1) < \varepsilon\big(\theta_2 - \theta_1\big) \longrightarrow 0 \ as\ \varepsilon \longrightarrow 0$$

Since $\varepsilon \longrightarrow 0$ and consequently R $\longrightarrow \infty$

Therefore, we get $\lim_{R \to \infty} \int_C f(z)dz$ = iA $\big(\theta_2 - \theta_1\big)$

NOTE:

1) In particular, if A = 0 then $\lim\limits_{R \to \infty} \int_C f(z)dz$ = 0.

2) Res f(z) = $\lim\limits_{z \to \infty}$ -z f(z)

Jordan's lemma 20:

If $f(z)$ is analytic except at finite number of singularities and if $f(z) \to 0$ as $z \to \infty$. Then $\lim_{R\to\infty} \int_\Gamma e^{imz} f(z)dz = 0$ $(m > 0)$ where ρ denotes the semi circle $|z| = R, \quad Im(z) > 0.$

Proof:

Choose R so large that all the singularities of f(z) lie with in Γ and none on its boundary. Since $f(z) \to 0$ uniformly as $|z| \to \infty$. Therefore, there exists $\varepsilon > 0$ such that $|f(z)| < \varepsilon$ for every z on Γ.

Now, $\left|\int_{\Gamma} e^{imz} f(z) dz\right| \leq \int_{\Gamma} \left|e^{imz}\right| |f(z)| |dz|$

Put $z = Re^{i\theta}$

$$< \int_0^{\pi} \left|e^{imRe^{i\theta}}\right| \varepsilon \left|Re^{i\theta} i\, d\theta\right|$$

$$= \varepsilon \int_0^{\pi} \left|e^{imR(cos\theta + isin\theta)}\right| R\, d\theta$$

$$= \varepsilon R \int_0^{\pi} \left|e^{imRcos\theta}\right| \left|e^{-mRsin\theta}\right| d\theta$$

$$= \varepsilon R \int_0^{\pi} e^{-mRsin\theta}\, d\theta \quad [\because \left|e^{imcos\theta}\right| = 1]$$

$$\leq \varepsilon R \int_0^{\pi} e^{-\left(mR\frac{2\theta}{\pi}\right)} d\theta \quad [\because \frac{2\theta}{\pi} \leq sin\theta \leq \theta]$$

$$= \frac{\varepsilon R\pi}{-mR2} \left[e^{-mR\frac{2\theta}{\pi}}\right]_0^{\pi}$$

$$= \frac{\pi R\varepsilon}{2mR} (1 - e^{-2mR})$$

$\to 0$ as $R \to \infty$ and $\varepsilon \to 0$

$$\lim_{R\to\infty} \int_{\Gamma} e^{imz} f(z) dz = 0$$

Result:

1. If $\lim_{z\to\infty} zf(z) = 0$, then $\lim_{R\to\infty} \int_{\Gamma} f(z) dz = 0$.

2. If $\lim_{R\to\infty} zf(z) = 0$, then $\lim_{R\to\infty} \int_{\Gamma} e^{imz} f(z) dz = 0$.

3. If C is the arc of the circle $|z-a|<r$ such that $\theta_1 \le \theta \le \theta_2$ and $\lim_{z\to a}(z-a)f(z)=A$ then $\lim_{r\to 0}\int_C f(z)dz = i\,A(\theta_1-\theta_2)$

4. If C is the arc of the circle $|z|=R$ such that $\theta_1 \le \theta \le \theta_2$ and $\lim_{R\to\infty} zf(z)=A$ then $\lim_{R\to\infty}\int_C f(z)dz = i\,A(\theta_2-\theta_1)$

TYPE II:

Evaluation of integrals of the form $\int_{-\infty}^{\infty} f(x)dx$,where $f(z)$ is analytic in the upper half plane except at a finite number of points and have no poles on the real axis.

Above type of integrals are evaluated by integrating $f(z)$ around C consisting of a semi circle Γ of radius R large enough to include all the poles of $f(z)$ and the part of the real axis from –R to R.

Therefore, by Cauchy's Residue theorem

$$\int_C f(z)dz = 2\pi i \times \text{(Sum of the residue of the poles of f(z)within C)}$$

$$\Rightarrow \int_{-R}^{R} f(x)dx + \int_{\Gamma} f(z)dz = 2\pi i \sum R^+$$

where $\sum R^+$ denotes the sum of the residue of the poles in the upper half.

It can be shown that, if $\lim_{z\to\infty} z\,dz = 0 \Rightarrow \lim_{R\to\infty}\int_{\Gamma} f(z)dz = 0$

Also, $\lim_{R\to\infty}\int_{-R}^{R} f(x)dx = \int_{-\infty}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum R^+$.

Problem 1: Evaluate $\int_0^{\infty} \frac{dx}{(1+x^2)^2}$

Solution:

Consider $\int_c f(z)dz$, where C is the contour consisting of a large semi circle Γ of radius R along with the part of the real axis from $-R\ to\ R,\ f(z) = \frac{1}{(1+z^2)^2}$

$$\therefore \int_C f(z)dz = 2\pi i \sum R^+,$$

where $\sum R^+ =$ Sum of the residue of the poles in the upper half plane

$$\int_{-R}^{R} f(x)dx + \int_{\Gamma} f(z)dz = 2\pi i \sum R^+$$

$$\lim_{z\to\infty} z\, f(z) = \lim_{z\to\infty} \frac{z}{(1+z^2)^2} = 0. and\ \lim_{R\to\infty} \int_{\Gamma} f(z)dz = 0$$

Also, $\lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \int_{-\infty}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum R^+$

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\pi i \sum R^+$$

To find the pole of $f(z)$,

$f(z) = \frac{1}{(1+z^2)^2}$. $\quad Now,\ 1 + z^2 = 0 \ \Rightarrow z = \pm i$

Therefore, $f(z)$ has only one pole z = i of order 2 lies in the upper half of the plane.

$$\underset{z=i}{Res} f(z) = \lim_{z\to i} \frac{1}{1!}\ \frac{d}{dz}((z-i)^2 f(z))$$

$$= \lim_{z\to i} \frac{d}{dz}\ (z-i)^2\ \frac{1}{(z+i)^2(z-i)^2}$$

$$\left[\because f(z) = \frac{1}{(1+z^2)^2} = \frac{1}{(z+i)^2(z-i)^2}\right]$$

$$= \lim_{z\to i} \frac{-2}{(z+i)^3}$$

$$= \frac{-2}{(2i)^3} = \frac{-2}{8i^3} = \frac{-1}{-4i} = \frac{1}{4i}$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\pi i \sum R^+ = 2\pi i \times \frac{1}{4i} = \frac{\pi}{2}$$

$$\therefore \int_{0}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$$

Problem 2. Evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)^3}$ (a is real and a > 0)

Solution: Here $f(z) = \frac{z^2}{(z^2+a^2)^3}$ and $\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \frac{z^3}{(z^2+a^3)^3} = 0$…(1)

Consider $\int_C f(z)dz$ where C is the contour consisting of a large semi circle

Γ of radius R along with the part of the real axis from $-R$ to R.

$$\therefore \int_C f(z)dz = 2\pi i \sum R^+$$

$$\int_{-R}^{R} f(x)dx + \int_{\Gamma} f(z)dz = 2\pi i \sum R^+$$

$$\int_{-R}^{R} \frac{x^2 dx}{(x^2+a^2)^3} + \int_{\rho} \frac{z^2 dz}{(z^2+a^2)^3} = 2\pi i \sum R^+$$

From (1), We have $\lim_{R\to\infty}\int_{\Gamma} f(z)dz = 0$

$$\Rightarrow \lim_{R\to\infty}\int_{\Gamma}\frac{z^2dz}{(z^2+a^2)^3} = 0 \_\_\_\_\_\_\_\_\_\_ (1)$$

Also, $\lim_{R\to\infty}\int_{-R}^{R} f(x)dx = \int_{-\infty}^{\infty} f(x)dx$

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x^2dx}{(x^2+a^2)^3} = \int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+a^2)^3}$$

$$\therefore \int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+a^2)^3} = 2\pi i\sum R^+$$

To find the pole of $f(z)$

$f(z) = \frac{z^2}{(z^2+a^2)^3} \Rightarrow z^2 + a^2 = 0 \Rightarrow z = \pm ai$

Poles of $f(z)$ are $z = \pm ai$

Therefore, $z = ai$ is the only pole of order 3 lies in the upper half plane.

$$\underset{z=ai}{Res} f(z) = \lim_{z\to ai}\frac{1}{2!}\frac{d^2}{dz^2}\left((z-ai)^3\frac{z^2}{(z^2+a^2)^3}\right)$$

$$= \lim_{z\to ai}\frac{1}{2!}\frac{d^2}{dz^2}\left((z-ai)^3\frac{z^2}{(z+ai)^3(z-ai)^3}\right)$$

$$= \lim_{z\to ai}\frac{1}{2}\frac{d^2}{dz^2}\left(\frac{z^2}{(z+ai)^3}\right)$$

$$= \lim_{z\to ai}\frac{1}{2}\frac{d}{dz}\left(\frac{2azi - z^2}{(z+ai)^4}\right)$$

$$= \lim_{z\to ai} \frac{1}{2}\left(\frac{-8aiz + 2z^2 - 2\,a^2}{(z+ai)^5}\right)$$

$$= \frac{1}{2}\left(\frac{-8ai(ai) + 2(-a^2) - 2\,a^2}{(2ai)^5}\right)$$

$$= \frac{1}{2}\ \frac{4a^2}{32a^5 i} = \ \frac{1}{16a^3 i}$$

Therefore, $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)^3} = 2\pi i\ \frac{1}{16a^3 i} = \frac{\pi}{8a^3}$

Problem 3. Evaluate $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9} dx$

Solution: Here f(z) = $\frac{z^2-z+2}{z^4+10z^2+9}$

Consider $\int_c f(z)dz$ where C is the contour consisting of a large semi circle Γ of radius R along with the part of the real axis from $-R \ to \ R$,

$$\therefore \int_C f(z)dz = 2\pi i \sum R^+$$

where $\sum R^+$ = sum of the residue of the poles in the upper half plane

$$\int_{-R}^{R} f(x)dx + \int_{\Gamma} f(z)dz = 2\pi i \sum R^+$$

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} z\left(\frac{z^2 - z + 2}{z^4 + 10z^2 + 9}\right) = 0$$

$$\text{Therefore,} \ \lim_{R\to\infty} \int_{\Gamma} f(z)dz = 0$$

Also, $\lim_{R\to\infty}\int_{-R}^{R} f(x)dx = \int_{-\infty}^{\infty} f(x)dx$

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum R^+$$

$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} = 2\pi i \sum R^+$$

To find the poles of $f(z) = \frac{z^2-z+2}{z^4+10z^2+9}$

$$z^4 + 10z^2 + 9 = 0$$

$$z^2 = \frac{-10 \pm \sqrt{100 - 4 \times 9}}{2}$$

$$= \frac{-10 \pm \sqrt{64}}{2} = \frac{-10 \pm 8}{2}$$

$$z^2 = -1, -9$$

$$z = i, 3i$$

$z = i$ is a pole of order 1 lies with in the upper half plane.

$$\underset{z=i}{Res} f(z) = \lim_{z\to i}(z - i)\, f(z)$$

$$= \lim_{z\to i}(z - i)\frac{z^2-z+2}{z^4+10z^2+9}$$

$$= \lim_{z\to i}(z - i)\frac{z^2-z+2}{(z-i)(z^3+9z+i(z^2+9))}$$

$$= \frac{-1-i+2}{-i+9i-i+9i} = \frac{1-i}{16i}$$

$z = 3i$ is a pole of order 1 lies with in the upper half plane

$$\operatorname*{Res}_{z=3i} f(z) = \lim_{z\to 3i}(z-3i)\frac{z^2-z+2}{(z-3i)[z^3+z+i(3z^2+3)]}$$

$$= \lim_{z\to 3i}\frac{-9-3i+2}{27i+3i-24i}$$

$$= \frac{-7-3i}{-48i} = \frac{7+3i}{48i}$$

$$\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9} = 2\pi i \times \left[\frac{1-i}{10} + \frac{7+3i}{48i}\right]$$

$$= 2\pi i \times \left[\frac{3-3i+7+3i}{48i}\right] \quad = 2\pi i \times \frac{10}{48i} \quad = \frac{5\pi}{12}$$

$$\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9} = \frac{5\pi}{12}$$

Problem 4. Evaluate $\int_0^\infty \frac{dx}{x^4+a^4} \quad (a > 0)$

Solution:

Consider $\int_C f(z)dz$, where C is the contour consisting of a large semi circle Γ of radius R. along with the part of the real axis from $-R$ to R,

$$\text{Therefore,} \int_C f(z)dz = 2\pi i \sum R^+$$

where $\sum R^+ =$ sum of the residue of the poles in the upper half plane

$$\int_{-R}^{R} f(x)dx + \int_{\Gamma} f(z)dz = 2\pi i \sum R^+$$

$$\int_{-R}^{R} \frac{dx}{x^4 + a^4} + \int_{\Gamma} \frac{dz}{z^4 + a^4} = 2\pi i \sum R^+$$

$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \frac{z}{z^4+a^4} = 0 \Rightarrow \lim_{R\to\infty} \int_\Gamma \frac{dz}{z^4+a^4} = 0$

Also, $\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^4+a^4} = \int_{-\infty}^{\infty} \frac{dx}{x^4+a^4}$. Therefore, $\int_{-\infty}^{\infty} \frac{dx}{x^4+a^2} = 2\pi i \sum R^+$

To find the pole of $f(z) = \frac{1}{z^4+a^4}$

$z^4 + a^4 = 0 \Rightarrow z^4 = -a^4 \Rightarrow z = (-1)^{\frac{1}{4}} a$

$$z = a\left(cos\frac{(2n+1)\pi}{4} + isin\frac{(2n+1)\pi}{4}\right) \text{ where } n = 0,1,2,3$$

Therefore, the poles are $ae^{i\pi/4}, ae^{i3\pi/4}, ae^{i5\pi/4}, ae^{i7\pi/4}$

Therefore, $ae^{i\pi/4}$ and $ae^{i3\pi/4}$ are the only poles with in C.

Let $\propto = \mathrm{a}e^{\frac{i\pi}{4}} \Rightarrow \alpha^4 = a^4 e^{i\pi} = -a^4$

$\mathrm{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha} (z-\alpha)\frac{1}{z^4+a^4} = \lim_{z\to\alpha} \frac{1}{4z^3}$

$= \lim_{z\to\alpha} \frac{1}{4\alpha^3} = \frac{\alpha}{4\alpha^4} = -\frac{\alpha}{4\alpha^4} = -\frac{\mathrm{a}e^{\frac{i\pi}{4}}}{4a^4} = -\frac{e^{\frac{i\pi}{4}}}{4a^3}$

Similarly, $\mathrm{Res}_{z=ae^{i3\pi/4}} f(z) = -\frac{e^{\frac{i3\pi}{4}}}{4a^3}$

$\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} = 2\pi i \sum R^+ = \frac{-2\pi i}{4a^3}\left(e^{\frac{i\pi}{4}} + e^{\frac{i3\pi}{4}}\right)$

$= \frac{-\pi i}{2a^3}\left(e^{\frac{i\pi}{4}} - e^{\frac{-i\pi}{4}}\right) = \frac{-\pi i}{2a^3}\left(2i\sin\frac{\pi}{4}\right)$

$= \frac{\pi}{a^3}(\sin\frac{\pi}{4}) \qquad = \frac{\pi}{\sqrt{2}a^3}$

$\int_0^{\infty} \frac{dx}{x^4+a^4} \quad = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{x^4+a^4}$

$= \frac{1}{2}\frac{\pi}{\sqrt{2}a^3} \qquad = \frac{\sqrt{2}\pi}{4a^3}$

Type III :

Evaluation of the integrals of the form $\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \sin mx$ dx , $\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \cos mx$ dx (m > 0) where (i) P(x) and Q(x) are polynomials (ii). Degree of Q(x) exceeds that of P(x) (iii). Q(x) has no real roots.

Above type of the integrals are evaluated by integrating $\int_C e^{imz}$ f(z) dz where f(z) = $\frac{P(z)}{Q(z)}$ around a contour C consisting of a semicircle Γ of radius R large enough to include all the poles of the integrand in the upper half plane and also part of the real axis from –R to +R

By Cauchy's residue theorem, $\int_C e^{imz}$f(z) dz = 2πi$\sum R^+$

$\int_{-R}^{+R} e^{imx}$f(x) dx + $\int_\Gamma e^{imz}$f(z) dz= 2πi$\sum R^+$

By Jordan's lemma, $\lim_{R \to \infty} \int_\Gamma e^{imz}$f(z) dz = 0 (m > 0)

[$as\ z \to \infty$ implies f(z) = $\frac{P(z)}{Q(z)} \to 0$]

$Also \lim_{R \to \infty} \int_{-R}^{+R} e^{imx}$f(x) dx = $\int_{-\infty}^{+\infty} e^{imx}$f(x) dx

$\int_{-\infty}^{\infty} e^{imx}$f(x) dx = $2\pi i \sum R^+$

$\int_{-\infty}^{\infty}(\cos mx + i \sin mx)$f(x) dx = $2\pi i \sum R^+$

$\int_{-\infty}^{\infty} \cos mx$ f(x) dx + i $\int_{-\infty}^{\infty} \sin mx$ f(x) dx = $2\pi i \sum R^{+}$

Equating the real and imaginary roots, we get the values of the given integral.

***Problem* 1** $\int_{0}^{\infty} \frac{\cos x}{x^2+a^2}$ dx, a real and $\int_{0}^{\infty} \frac{x \sin x}{x^2+a^2}$ dx , a real

Solution:

Consider $\int_C \frac{e^{imz}}{z^2+a^2}$dz = $\int_C e^{imz}$ f(z) dz where C is the contour consisting of a semicircle of radius R large enough to include all the poles of f(z) in the upper half plane and also the part of the real axis from $-R$ to $+R$.

Therefore, By Cauchy's residue theorem,

$\int_C e^{imz}$f(z) dz = $2\pi i \sum R^{+}$

$\int_{-R}^{R} e^{imx}$f(x) dx + $\int_\Gamma e^{imz}$ f(z) dz = $2\pi i \sum R^{+}$

$\int_{-R}^{R} \frac{e^{imx}}{x^2+a^2}$dx + $\int_\Gamma \frac{e^{imz}}{z^2+a^2}$ dz = $2\pi i \sum R^{+}$

Since $\lim\limits_{z \to \infty}$f(z) = $\lim\limits_{z \to \infty} \frac{1}{z^2+a^2}$ = 0. By Jordan lemma, $\lim\limits_{R \to \infty} \int_\Gamma \frac{e^{imz}}{z^2+a^2}$dz = 0

Also $\lim\limits_{R \to \infty} \int_{-R}^{+R} \frac{e^{imx}}{x^2+a^2}$ dx = $\int_{-\infty}^{+\infty} \frac{e^{imx}}{x^2+a^2}$ dx

Therefore, $\int_{-\infty}^{+\infty} \frac{e^{imx}}{x^2+a^2}$dx = $2\pi i \sum R^{+}$

To find the poles of f(z)

Z = $\pm ai$ are the poles of f(z) and Z = ai is the only pole lies within C.

$\underset{z = ai}{Res}$f(z) = $\lim\limits_{z \to ai}$(z - ai) f(z)

$= \lim_{z \to ai} (z - ai) \frac{e^{imz}}{(z-ai)(z+ai)} \quad = \frac{e^{-ma}}{2ai}$

$\int_{-\infty}^{+\infty} \frac{e^{imx}}{x^2+a^2} dx = 2\pi i \sum R^{+} \; = 2\pi i \frac{e^{-ma}}{2ai} \quad = \frac{\pi}{a} e^{-ma}$

$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2} dx + i\int_{-\infty}^{\infty} \frac{\sin mx}{x^2+a^2} \, dx \; = \; \frac{\pi}{a} e^{-ma}$

Equating real and imaginary parts, $\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2} dx = \frac{\pi}{a} e^{-ma} \rightarrow (1)$

$\int_{-\infty}^{\infty} \frac{\sin mx}{x^2+a^2} \, dx \; = 0$

When m=1, $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} \, dx = \frac{\pi}{a} e^{-a}$

$\int_{0}^{\infty} \frac{\cos x}{x^2+a^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} \, dx = \frac{1}{2} \frac{\pi}{a} e^{-a} = \frac{\pi}{2a} e^{-a}$

Differentiate (1) w.r.to m, $\int_{-\infty}^{\infty} - \frac{\sin mx}{x^2+a^2} \, x dx$

$= \frac{\pi}{a} e^{-ma} (-a)$

$= -\pi e^{-ma}$

Put m = 1, $-\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2} dx = -\pi e^{-a}$

$\int_{0}^{\infty} \frac{x \sin x}{x^2+a^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2} dx \quad = \frac{1}{2} \pi e^{-a}$

Problem 2. $\int_{0}^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}$

Solution:

Consider $\int_C f(z) dz$, where f(z) = $\frac{e^{imz}}{z}$. It has a singularity at z = 0 on the real axis . Let the contour C consists of large semicircle $|z| = R$ indented at z = 0 and $\in$ be the radius of this small semicircle of identation. Now there is no singularities within C.

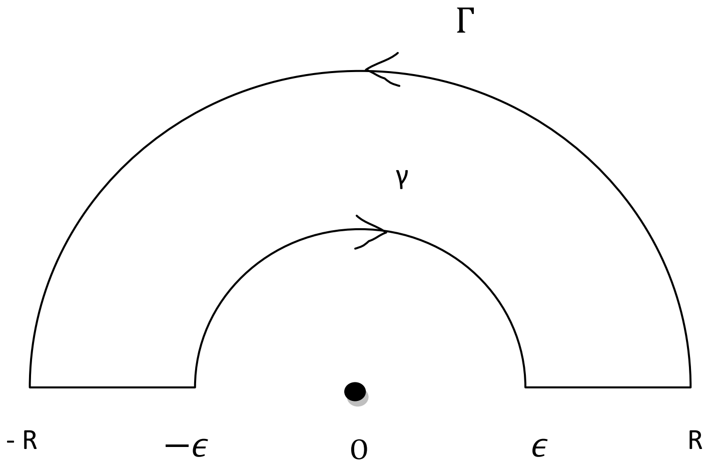

Fig.5.2

By Cauchy's residue theorem, $\int_c f(z)dz = 2\pi i \sum R^+ = 0$

$$\int_{-R}^{-\epsilon} f(z)dz + \int_{\gamma} f(z)dz + \int_{\epsilon}^{R} f(z)dz + \int_{\Gamma} f(z)dz = 0$$

$$\lim_{R \to \infty} \int_{\rho} e^{imz} g(z) = 0 \text{ , since } \lim_{z \to \infty} g(z) = \lim_{z \to \infty} \frac{1}{z} = 0$$

$$(i.e) \lim_{R \to \infty} \int_{\Gamma} \frac{e^{imz}}{z} = 0 \text{ [By Jordan lemma]}$$

Consider $\int_{\gamma} f(z)dz$, where γ is the circle $|z| = \in$

$$\lim_{z \to 0} zf(z) = \lim_{z \to 0} z \frac{e^{imz}}{z} = 1$$

$$\lim_{\in \to 0} \int_{\gamma} f(z)dz = -iA(\theta_2 - \theta_1) = -i(\pi - 0) = -i\pi$$

The negative sign is taken because the orientation is clockwise

Taking R$\to \infty$ and $\in \to 0$,

$\int_{-\infty}^{0} f(z)dz \ - i\pi + \int_{0}^{\infty} f(z)dz + 0 = 0 \ \Rightarrow \int_{-\infty}^{\infty} f(z)dz = i\pi$

$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{imz}}{z} dz = i\pi \ \Rightarrow \ \int_{-\infty}^{+\infty} \left(\frac{\cos mz + i \sin mz}{z}\right) dz = i\pi$

Equating real and imaginary parts, $\int_{-\infty}^{+\infty} \frac{\cos mz}{z} dz = 0$ and $\int_{-\infty}^{+\infty} \frac{\sin mz}{z} dz = \pi$

Therefore, $2\int_{0}^{\infty} \frac{\sin mz}{z} dz = \ \pi \Rightarrow \int_{0}^{\infty} \frac{\sin mz}{z} dz = \ \frac{\pi}{2}$.

Type IV :

Evaluation of the integrals of the type $\int_{0}^{\infty} x^{\alpha} f(x)dx$

Problem 1: $\int_{0}^{\infty} \frac{x^{a-1}}{1+x^2} dx$ and hence deduce that $\int_{0}^{\infty} \frac{(\log x)^2}{1+x^2} dx = \ \frac{\pi^3}{8}$ and

$$\int_{0}^{\infty} \frac{x^{a-1} - x^{b-1}}{\log x} \frac{dx}{1+x^2} = \log \frac{tan \frac{\pi a}{4}}{tan \frac{\pi b}{4}}$$

Solution: Consider $\int_{C} f(z)dz$, where f(z) $= \frac{z^{a-1}}{1+z^2}$, 0 < a < 2. z = 0 is a singularity of f(z) for 0 < a < 1.

The poles of f(z) are given by z = $\pm i$ $[z^2 + 1 = 0]$

The contour C consists of the large semicircle $|z|$ = R in the upper half plane and the real axis from $-R$ to $+R$ intented at z = 0 and by a small circle ɣ of radius ϵ, the only pole lying within C is z = i.

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z\to i} (z-i)f(z)$$

$$= \lim_{z\to i} (z-i) \frac{z^{a-1}}{(z+i)(z-i)} = \lim_{z\to i} \frac{z^{a-1}}{(z+i)}$$

$$= \frac{i^{a-1}}{2i} = \frac{-i.i^{a-1}}{2} = \frac{-1}{2} i^a = \frac{-1}{2} e^{\frac{\pi i a}{2}} \qquad [\text{as } i = e^{\frac{\pi i}{2}}]$$

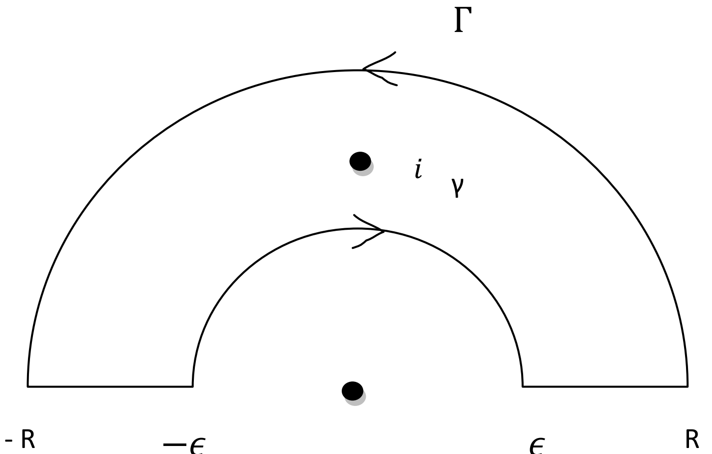

Fig.5.3

By Cauchy's Residue Theorem, $\int_C f(z)dz = 2\pi i\Sigma R^+$

$$\int_{-R}^{-\epsilon} f(z)dz + \int_{\gamma} f(z)dz + \int_{\epsilon}^{R} f(z)dz + \int_{\Gamma} f(z)dz$$

$$= 2\pi i \left(\frac{-1}{2}\right) e^{\frac{\pi i a}{2}} = -\pi i e^{\frac{\pi i a}{2}} \text{ ....(1)}$$

$$\lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{z.z^{a-1}}{z^2+1} = 0$$

$\implies \lim_{\epsilon\to 0} \int_{\gamma} f(z)dz = -i \times 0 \times (\pi - 0) = 0$

Also, $\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} z\frac{z^{a-1}}{1+z^2} = \lim_{Z\to\infty} \frac{z^a}{1+z^2}$

$$= \lim_{z\to\infty} \frac{1}{z^{2-a}\,(1+\frac{1}{z^2})} = 0 \quad [\,as, 2-a > 0]$$

$\therefore \lim_{R\to\infty} \int_{\Gamma} f(z)dz = 0$

Taking Limit $\epsilon \to 0$ and R$\to \infty$ in (1), We have

$$\int_{-\infty}^{0} f(z)dz + 0 + \int_{0}^{\infty} f(z)dz + 0 = -\pi i e^{\frac{\pi i a}{2}}$$

$\int_{-\infty}^{0} \frac{x^{a-1}}{1+x^2} dx + \int_{0}^{\infty} \frac{x^{a-1}}{1+x^2} dx = -\pi i e^{\frac{\pi i a}{2}}$ …..(2)

Consider $\int_{-\infty}^{0} \frac{x^{a-1}}{1+x^2} dx$

Put x = -y ; x = -∞ ; y = ∞ ; x = 0 ; y = 0

$$\text{Therefore,} I = \int_{\infty}^{0} \frac{(-y)^{a-1}}{1+y^2}(-dy) = \int_{0}^{\infty} \frac{(-1)^{a-1}y^{a-1}}{1+y^2} dy$$

$= -\int_{0}^{\infty} \frac{(-1)^a y^{a-1}}{1+y^2} dy = -e^{\pi i a} \int_{0}^{\infty} \frac{y^{a-1}}{1+y^2} dy$ [-1=$e^{\pi i}$]

Substitute in (2), $-e^{\pi i a} \int_{0}^{\infty} \frac{x^{a-1}}{1+x^2} dx + \int_{0}^{\infty} \frac{x^{a-1}}{1+x^2} dx = -\pi i e^{\frac{\pi i a}{2}}$

$$\left(1 - e^{\pi i a}\right) \int_{0}^{\infty} \frac{x^{a-1}}{1+x^2} dx = -\pi i e^{\frac{\pi i a}{2}}$$

$$e^{\frac{\pi i a}{2}} \left(e^{\frac{-\pi i a}{2}} - e^{\frac{\pi i a}{2}}\right) \int_{0}^{\infty} \frac{x^{a-1}}{1+x^2} dx = -\pi i e^{\frac{\pi i a}{2}}$$

$$-2\mathrm{i} \sin\frac{\pi a}{2} \int_{0}^{\infty} \frac{x^{a-1}}{1+x^2} dx = -\pi i$$

$\therefore \int_{0}^{\infty} \frac{x^{a-1}}{1+x^2} dx = \frac{\pi}{2} \operatorname{cosec}\frac{\pi a}{2}$……(3)

Diff. w.r.t. a, $\int_{0}^{\infty} \frac{x^{a-1}\log x}{1+x^2} dx = \frac{-\pi^2}{4} \operatorname{cosec}\frac{\pi a}{2} \cot\frac{\pi a}{2} = \frac{-\pi^2}{4} \frac{cos\frac{\pi a}{2}}{sin^2\frac{\pi a}{2}}$

Diff. again w.r.t. a $\int_0^\infty \frac{x^{a-1}\,(\log x)^2}{1+x^2}dx = \frac{\pi^3}{8}\frac{1+cos^2\frac{\pi a}{2}}{sin^2\frac{\pi a}{2}}$

Put a = 1, $\int_0^\infty \frac{(\log x)^2}{1+x^2}dx = \frac{\pi^3}{8}\frac{1+cos^2\frac{\pi}{2}}{sin^2\frac{\pi}{2}} = \frac{\pi^3}{8}$

Integrating (3) w.r.t. a,

$$\int_0^\infty \frac{x^{a-1}}{\log x(1+x^2)}dx = \left(\frac{\pi}{2}\right)\frac{\log\tan\frac{\pi a}{4}}{\left(\frac{\pi}{2}\right)} = \log\tan\frac{\pi a}{4} \qquad \text{.....(4)}$$

Also $\int_0^\infty \frac{x^{b-1}}{\log x(1+x^2)}dx = \log\tan\frac{\pi b}{4}$ .....(5)

Now, (4) - (5)

$$\int_0^\infty \frac{x^{a-1} - x^{b-1}\ dx}{(\log x)(1+x^2)} = \log\tan\frac{\pi a}{4} - \log\tan\frac{\pi b}{4}$$

$$= \log\left(\frac{tan\frac{\pi a}{4}}{tan\frac{\pi b}{4}}\right)$$

Problem 2. $\int_0^\infty \frac{\log(1+x^2)\ (dx)}{(x)^{1+\alpha}}$, $0 < \alpha < 1$

Solution:

Consider $\int_c f(z)dz$ where f(z) = $\frac{\log(1+z^2)}{(z)^{1+\alpha}}$ where C is the semi-circle given by |z| =R where R is every large intended at z = 0 by a small semi-circle γ of radius $\in$ and the real axis from –R to R.

Therefore, f(z) is analytic with C.

Therefore, By Cauchy residue theorem, $\int_c f(z)dz = 0$

$$\int_{-R}^{-\in} f(z)dz + \int_\gamma f(z)dz + \int_\in^R f(z)dz + \int_\Gamma f(z)dz = 0 \qquad \text{........(1)}$$

$\lim_{z\to 0} zf(z) \;=\; \lim_{z\to 0} z\,\frac{\log(1+z^2)}{z^{1+\alpha}}$

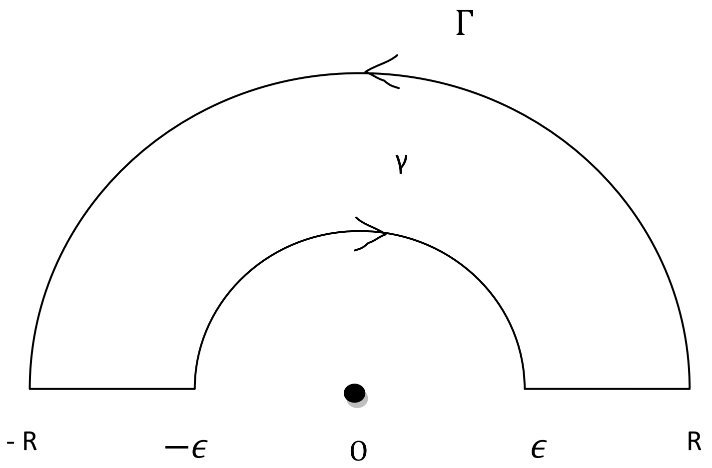

Fig.5.4

$$= \lim_{z\to 0} \frac{\left(z^2 - \frac{z^4}{2} + \ldots\ldots\ldots\right)}{z^{\alpha}} \text{ [Since, |z|<1]}$$

$$= \;\; 0$$

$$\Longrightarrow \lim_{\epsilon\to 0} \int_{\gamma} f(z)dz \quad = 0$$

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} z\,\frac{\log(1+z^2)}{z^{1+\alpha}}$$

$$= \lim_{z\to\infty} \frac{1}{z^{\alpha}} \log z^2 \left(1 + \frac{1}{z^2}\right)$$

$$= \lim_{z\to\infty} \frac{2\,log\,z + log\,\left(1+\frac{1}{z^2}\right)}{z^{\propto}}$$

$$= \lim_{z\to\infty} \left[2z^{1-\alpha}\,\frac{log\,z}{z} + \frac{1}{z^{\propto}}\,log\left(1+\frac{1}{z^2}\right)\right]$$

$$= 2\lim_{z\to\infty} z^{1-\alpha}\,\frac{log\,z}{z} + \lim_{z\to\infty} \frac{1}{z^{\propto}}\left(\frac{1}{z^2} - \frac{1}{2z^4} + \ldots\ldots\ldots\right)$$

$$= 0 \quad \left[\text{Since, } \lim_{z\to\infty} \frac{log\,z}{z} = 0\right]$$

$\lim\limits_{z\to\infty} \int_{\Gamma} f(z)dz \quad = 0$ [by lemma (2)]

Taking limit R $\to \infty$ and $\in \to 0$ in (1)

$\int_{-\infty}^{0} f(x)dx + 0 + \int_{0}^{\infty} f(x)dx + 0 = 0$

$\int_{-\infty}^{0} \frac{\log(1+x^2)\ dx}{(x)^{1+\alpha}} + \int_{0}^{\infty} \frac{\log(1+x^2)\ dx}{(x)^{1+\alpha}} = 0$ ………(2)

Consider I = $\int_{-\infty}^{0} \frac{\log(1+x^2)\ dx}{x^{1+\alpha}}$

Put x = -y When x= -∞ , y= ∞ and x= 0, y= 0

Therefore, I = $\int_{\infty}^{0} \frac{\log(1+y^2)\ (-dy)}{(-y)^{\alpha+1}}$

$= \int_{0}^{\infty} \frac{\log(1+y^2)\ dy}{(-y)(-y)^{\alpha}}$ [since $-1 = e^{\pi i}$]

$= \int_{0}^{\infty} \frac{\log(1+y^2)\ dy}{(-1)(-1)^{\alpha}(y)^{1+\alpha}}$

$= -\int_{0}^{\infty} \frac{\log(1+y^2)\ dy}{e^{i\pi\alpha}\, y^{1+\alpha}}$

$= -e^{\pi i\alpha} \int_{0}^{\infty} \frac{\log(1+x^2)\ dx}{x^{1+\alpha}}$ …………… (3)

Sub (3) in (2)

$0 = e^{-\pi i\alpha} \int_{0}^{\infty} \frac{\log(1+x^2)\ dx}{x^{1+\alpha}} + \int_{0}^{\infty} \frac{\log(1+x^2)\ dx}{x^{1+\alpha}}$

Therefore, $(1- e^{\pi i\alpha}) \int_{0}^{\infty} \frac{\log(1+x^2)\ dx}{x^{1+\alpha}} = 0$

Therefore, $\int_{0}^{\infty} \frac{\log(1+x^2)\ dx}{x^{1+\alpha}} = 0$

Problem 3. Evaluate $\int_0^{\infty} \frac{\log(x)\ dx}{1+x^2}$ and $\int_0^{\infty} \frac{(\log x)^2\ dx}{1+x^2}$

Solution:

Consider $\int_c f(z)dz$, where f(z) = $\frac{(log\ z)^2}{1+z^2}$, z = 0 the branch point of f(z)

and poles of f (z) is given by z = +i and z = -i.

Let C be the large semi-circle, |z| = R indented at z = 0 and ∈ the radius of small semi-circle of indentation. The pole z = i lies within C.

$$\text{Res}_{z=i} f(z) = \lim_{z\to i}(z-i)\ \frac{(log\ z)^2}{(z+i)(z-i)}$$

$$= \frac{(log\ i)^2}{2i} = \frac{(\ i\frac{\pi}{2})^2}{2i} = -\frac{\pi^2}{8i} \quad [\text{Since } \log i = i\frac{\pi}{2}\ ]$$

Therefore, By Cauchy Residue Theorem,

Therefore, $\int_C f(z)dz = 2\,\pi\ \text{i} \sum R^{+}$

$$= 2\,\pi\ \text{i}\ (-\frac{\pi^2}{8i}) = -\frac{\pi^3}{4}$$

$$\int_{-R}^{-\epsilon} f dz + \int_{\gamma} f(z)dz + \int_{\epsilon}^{R} f(z)dz + \int_{\Gamma} f(z)dz = 0 \quad \ldots\ldots(1)$$

$$\lim_{z\to\infty} z\, f(z) = \lim_{z\to\infty} \frac{z(log\ z)^2}{1+z^2} = \lim_{z\to\infty} \frac{z(log\ z)^2}{z^2(1+\frac{1}{z^2})}$$

$$= \lim_{z\to\infty} \frac{(log\ z)^2}{z} \lim_{z\to\infty} \frac{1}{(1+\frac{1}{z^2})}$$

$$= \lim_{z\to\infty} \frac{(log\ z)^2}{z} \left(\frac{\infty}{\infty}\right) = \lim_{z\to\infty} \frac{2\ log\ z}{z}$$

$$= 2\ (\ 0) \qquad [\ \text{Since}, \lim_{z\to\infty} \frac{log\ z}{z} = 0\ ]$$

$$= \quad 0$$

Therefore, $\lim\limits_{R\to\infty} \int_{\Gamma} f(z)dz = 0$

$\lim\limits_{z\to 0} z\, f(z) = \lim\limits_{z\to 0} \frac{z(\log z)^2}{1+z^2}$

Put $z = \frac{1}{t}$

Therefore, $\lim\limits_{z\to 0} z\, f(z) = \lim\limits_{t\to\infty} \frac{(\log \frac{1}{t})^2}{t(1+\frac{1}{t^2})} = \lim\limits_{t\to\infty} \frac{(-\log t)^2}{t(1+\frac{1}{t^2})}$

$= \lim\limits_{t\to\infty} \frac{t(\log t)^2}{1+t^2} = 0$ (as above)

Therefore, $\lim\limits_{\epsilon\to 0} \int_{\gamma} f(z)dz = 0$

Applying lim $\in \to 0$ and R $\to \infty$ we get

(1) $\Longrightarrow \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = -\pi^3/4$

$\int_{-\infty}^{0} \frac{(\log x)^2 dx}{1+x^2} + \int_{0}^{\infty} \frac{(\log x)^2 dx}{1+x^2} = \frac{-\pi^3}{4}$ ....(2)

Consider $I = \int_{-\infty}^{0} \frac{(\log x)^2 dx}{1+x^2}$

Put x = -y when x = -∞ , y = ∞ and x = 0 , y =0

$I = \int_{\infty}^{0} \frac{(\log(-y))^2(-dy)}{1+y^2}$

$= \int_{0}^{\infty} \frac{(\log(-1)+ \log y)^2 dy}{1+y^2}$

$I = \int_{0}^{\infty} \frac{(\pi i+ \log y)^2 dy}{1+y^2}$ [Since, $\log(-1) = \pi i$]

$$= \int_0^\infty \frac{-\pi^2 + (\log y)^2 + 2\pi i logy}{1+y^2} dy$$

$$= -\pi^2 \int_0^\infty \frac{dy}{1+y^2} + \int_0^\infty \frac{(\log y)^2 dy}{1+y^2} + 2\pi i \int_0^\infty \frac{\log y \, dy}{1+y^2}$$

$$= -\pi^2 (\tan^{-1} y)_0^\infty + \int_0^\infty \frac{(\log y)^2 dy}{1+y^2} + 2\pi i \int_0^\infty \frac{\log y \, dy}{1+y^2}$$

$$= \frac{-\pi^3}{2} + \int_0^\infty \frac{(\log y)^2 dy}{1+y^2} + 2\pi i \int_0^\infty \frac{\log y \, dy}{1+y^2} \ldots.(3)$$

From (2) and (3)

$$\frac{-\pi^3}{2} + \int_0^\infty \frac{(\log y)^2 dy}{1+y^2} + 2\pi i \int_0^\infty \frac{\log y \, dy}{1+y^2} + \int_0^\infty \frac{(\log y)^2 dy}{1+y^2} = \frac{-\pi^3}{4}$$

Equating Real and Imaginary Part

$$\frac{-\pi^3}{2} + 2 \int_0^\infty \frac{(\log y)^2 dy}{1+y^2} = \frac{-\pi^3}{4}$$

$$2 \int_0^\infty \frac{(\log y)^2 dy}{1+y^2} = \frac{-\pi^3}{4} + \frac{\pi^3}{2} = \frac{\pi^3}{4}$$

$$\int_0^\infty \frac{(\log y)^2 dy}{1+y^2} = \frac{\pi^3}{8} \text{ and } \int_0^\infty \frac{\log y \, dy}{1+y^2} = 0$$

TYPE: V Integrals involving many valued function:

Problem 1:

Evaluate $\int_0^\pi \log \sin x \, dx$

Solution:

Consider $\int_C f(z)dz$, where f (z) = log (1- e^{2iz}) and choose the contour C

as follows , within the contour C , f(z) is regular.

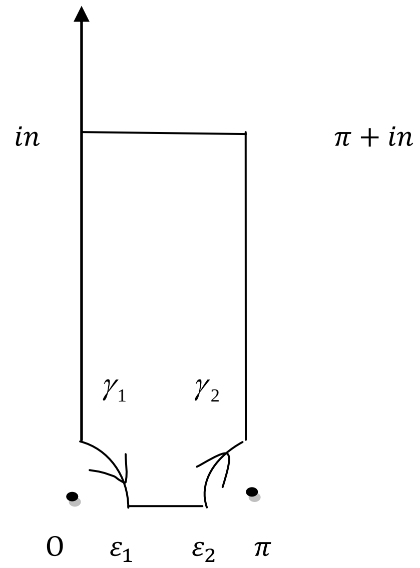

Fig.5.5

By Cauchy Residue Theorem, $\int_C f(z)dz = 0$, where f (z) = log (1- e^{2iz})

$$\Longrightarrow \quad \int_{\in 1}^{\pi - \in 2} f(z)dz + \int_{\gamma 2} f(z)dz + \int_{\in 2}^{n} f(\pi + iy)idy + \int_{\pi}^{0} f(x + in)dx$$

$$+ \int_{n}^{\in 1} f(iy)idy + \int_{\gamma 1} f(z)dz = 0$$

$$\Longrightarrow I_1 + I_2 + I_3 + I_4 + I_5 + I_6 = 0$$

Consider $I_3 = \int_{\in 2}^{n} f(\pi + iy)idy$

$$= \int_{\in 2}^{n} log\,(1 - e^{2i(\pi + iy)})idy$$

$$= \int_{\in 2}^{n} log\,(1 - e^{-2y})idy$$

$$I_5 = \int_{n}^{\in 1} f(iy)idy$$

$$= \int_{n}^{\in 1} log\,(1 - e^{2i(iy)})idy$$

$$= \int_{n}^{\in 1} log\,(1 - e^{-2y})idy$$

As n $\rightarrow \infty$, $\in_1, \in_2 \rightarrow 0$

$I_3 \rightarrow \int_0^\infty log\ (1 - e^{-2y}) idy$

$I_5 \rightarrow \int_\infty^0 log\ (1 - e^{-2y}) idy$ = - $\int_0^\infty log\ (1 - e^{-2y}) idy$

$I_3 + I_5 \rightarrow 0$

Therefore, $\lim\limits_{z \to 0}$ z f (z) = $\lim\limits_{z \to 0}$ z log $(1 - e^{2iz})$

$$= \lim_{z \to 0} \frac{log\ (1 - e^{i2z})}{\frac{1}{z}}$$

$$= \lim_{z \to 0} \frac{\frac{1}{1 - e^{i2z}}(-e^{i2z})2i}{-\frac{1}{z^2}}$$

$$= \lim_{z \to 0} \frac{2iz^2 e^{i2z}}{1 - e^{i2z}} \text{ (By L'Hopital rule)}$$

$$= \lim_{z \to 0} \frac{2ize^{i2z} + 2iz^2\ 2ie^{i2z}}{-2ie^{i2z}} = 0$$

$\lim_{\epsilon_1 \to 0} \int_{\gamma_1} f(z) dz = 0.$

Similarly, $\lim_{\epsilon_2 \to 0} \int_{\gamma_2} f(z) dz = 0$

Consider $I_4 = \int_\pi^0 log\ (1 - e^{2i(x+in)}) dx$

$$= \int_0^\pi log\ (1 - e^{2ix}\ e^{-2n}) dx \rightarrow 0 \text{ as } n \rightarrow \infty$$

As n $\rightarrow \infty$, $\in_1, \in_2 \rightarrow 0$ we get $\int_0^\pi f(x) dx = 0.$

$$\int_0^\pi log\ (1 - e^{2ix}) dx = 0$$

$1 - e^{2ix} = e^{ix} (e^{-ix} - e^{ix}) = e^{ix}$ (- 2i sinx)

$\int_0^{\pi} log\, e^{ix}\,(-\,2\text{i sinx})\, dx\; = 0$

$\Longrightarrow\; \int_0^{\pi} log2\, dx + \int_0^{\pi} \log(-i)\, dx + \int_0^{\pi} loge^{ix}\, dx + \int_0^{\pi} logsinx\, dx = 0$

$\Longrightarrow\; \pi\log2 + \log(\text{-i})\, \pi + \int_0^{\pi} ix\, dx + \int_0^{\pi} logsinx\, dx = 0$

$\Longrightarrow\; \pi\log2 - \frac{\pi^2 i}{2} \;+\; \left(\frac{x^2}{2}\right)_0^{\pi} + \int_0^{\pi} logsinx\, dx = 0$

$\Longrightarrow\; \pi\log2 - \frac{\pi^2 i}{2} \;+\; \frac{\pi^2 i}{2} + \int_0^{\pi} logsinx\, dx = 0$

$$\int_0^{\pi} logsinx\, dx \;=\; -\,\pi\; log2.$$